

FUNCTORIALITY OF THE BGG CATEGORY \mathcal{O}

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Dedicated to the memory of Israel Moiseevich Gelfand

ABSTRACT. This article aims to contribute to the study of algebras with triangular decomposition over a Hopf algebra, as well as the BGG Category \mathcal{O} . We study functorial properties of \mathcal{O} across various setups. The first setup is over a skew group ring, involving a finite group Γ acting on a regular triangular algebra A . We develop Clifford theory for $A \rtimes \Gamma$, and obtain results on block decomposition, complete reducibility, and enough projectives. \mathcal{O} is shown to be a highest weight category when A satisfies one of the “Conditions (S)” ; the BGG Reciprocity formula is slightly different because the duality functor need not preserve each simple module.

Next, we turn to tensor products of such skew group rings; such a product is also a skew group ring. We are thus able to relate four different types of Categories \mathcal{O} ; more precisely, we list several conditions, each of which is equivalent in any one setup, to any other setup - and which yield information about \mathcal{O} .

1. INTRODUCTION

The results of this article relate the representation theories of various algebras; thus, they are “functorial” in a sense. However, one can apply them to certain well-understood algebras, to get results in other setups. For example, we show the following result in Proposition 3.1 and after Remark 4.2; for details and more results in this setting, also look after Remark 4.2.

Proposition 1.1. *Given a complex semisimple Lie algebra \mathfrak{g} , denote by $P_{\mathfrak{g}}^+$ its set of dominant integral weights, and $R_{\mathfrak{g}}$ the wreath product algebra $S_n \wr \mathfrak{U}\mathfrak{g}$ (see [Mac, §6]). Then the category of finite-dimensional $R_{\mathfrak{g}}$ -modules contains at least “ $P_{\mathfrak{g}}^+$ -many” simple objects, and is semisimple.*

Hereafter, $S_n \wr A := A^{\otimes n} \rtimes S_n$ for any ring A , with S_n acting by permuting the components in the tensor product; the definition is similar when A is a group.

This article studies the representation theory of special families of algebras with triangular decomposition over a commutative Hopf algebra (these

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algebras have been studied in general by Bazlov and Berenstein). In general, such algebras are not Hopf algebras; thus one studies them, for example, by defining and analyzing (analogues of) Verma modules - or, in other words, some version of the Bernstein-Gelfand-Gelfand Category \mathcal{O} . We do so below, for a special subclass of such algebras.

In [Kh2], we defined a general framework of a *regular triangular algebra* A (also denoted by RTA ; we recall the definition below), wherein the BGG Category \mathcal{O} can be studied, and results obtained about a block decomposition into highest weight categories. Examples of such algebras are (quantized) universal enveloping algebras of (semisimple, or) symmetrizable Kac-Moody Lie algebras, Heisenberg and Virasoro algebras, and (quantized) infinitesimal Hecke algebras (see [Kh1, GK, EGG], and [KT, Kh2] respectively).

The goal of this article is to extend many of the classical results of [BGG] to other setups (e.g., Proposition 1.1 above), by applying the following two constructions:

- the *skew group ring* $A \rtimes \Gamma$ (where Γ is a finite group acting on A),
- the *tensor product* of RTAs A_1, \dots, A_n for some n .

These constructions were motivated by the study of finite-dimensional representations of the wreath product symplectic reflection algebra in [EM]; we term the first construction *Clifford theory*. Combining them produces results for the wreath product of an RTA, for instance.

We now combine the two constructions as follows: suppose $A_i \rtimes \Gamma_i$ are skew group rings for $1 \leq i \leq n$. We can then form $A = \otimes_{i=1}^n A_i$, $\Gamma = \times_{i=1}^n \Gamma_i$, and $A \rtimes \Gamma$ - and this gives us four different setups for the category \mathcal{O} :

$$\text{all } A_i = \{A_i : 1 \leq i \leq n\}, \text{ all } A_i \rtimes \Gamma_i = \{A_i \rtimes \Gamma_i : 1 \leq i \leq n\}, A, A \rtimes \Gamma. \quad (1.1)$$

In a sense, these constructions “commute” when $\Gamma = \times_i \Gamma_i$, namely:

$$\begin{array}{ccc} \{A_i\} & \xrightarrow{\times} & \{A_i \rtimes \Gamma_i\} \\ \otimes \downarrow & & \otimes \downarrow \\ A = \otimes_i A_i & \xrightarrow{\times} & A \rtimes \Gamma \end{array} \quad (1.2)$$

(Every diagram in this article is functorial, rather than a commuting square of morphisms in some categories.) However, if we want to construct the wreath product $S_n \wr A$, say, then the diagram above does not help: the steps to take are $\{A_i = A, |\Gamma_i| = 1\} \longrightarrow A^{\otimes n} \longrightarrow A^{\otimes n} \rtimes S_n$. But these steps are all found in diagram (1.2); moreover, all algebras here (as well as in (1.2)) are examples of skew group rings.

Our goal, therefore, is twofold: (a) to relate the categories \mathcal{O} in the above four setups, and (b) to show that $\mathcal{O}_{A \rtimes \Gamma}$ is a highest weight category in the sense of Cline, Parshall, and Scott [CPS], when $A \rtimes \Gamma$ (or A) satisfies what was called *Condition (S)* in [Kh1] - but we now call *Condition (S3)*, as in [Kh2]. As we will see, this is related to this condition being satisfied in the other three setups.

Getting results using the second vertical arrow in diagram (1.2) may pose problems - for instance, see diagram (16.1) below. However, the horizontal arrows can be “reversed”, which allows us to proceed the “longer” way in this case. Also note that much of the analysis will be analogous to the theory developed in [BGG, Kh1, GK]; however, we will explain the new features - as well as the interconnections - in detail.

If H is cocommutative as well, then skew group rings are algebras with triangular decomposition over the Hopf algebra $H \rtimes \Gamma$ (see [BaBe, Appendix]). Thus, one avenue for possible further study, is to bring the theory of braided doubles and triangular ideals into the picture.

Finally, as a small application, we study the wreath product of symplectic oscillator algebras; we conclude by showing that the Poincare-Birkhoff-Witt property does not hold if we deform certain relations.

2. SETUPS - THE GENERAL AND HOPF CASES

We work throughout over a ground field k . Unless otherwise specified, all tensor products are over k . Define $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$. Given $S \subset \mathbb{Z}$ and a finite subset Δ of an abelian group \mathcal{P}_0 , the symbols $(\pm)S\Delta$ stand for $\{(\pm) \sum_{\alpha \in \Delta} n_\alpha \alpha : n_\alpha \in S \forall \alpha\} \subset \mathcal{P}_0$. We will often abuse notation and claim that two modules or functors are equal, when they are isomorphic (double duals, for instance). Finally, in developing Clifford theory for RTAs and the Category \mathcal{O} , we use the general results on Clifford theory over \mathbb{C} , that are stated in the Appendix in [Mac].

Definition 2.1.

- (1) If A is a k -algebra, and Γ a group acting on A by k -algebra automorphisms, then the (Γ) -skew group ring over A is defined to be $A \rtimes \Gamma := A \otimes_k k\Gamma$, with relations $\gamma a = \gamma(a)\gamma$. (Henceforth, $k\Gamma$ is the group algebra of Γ .)
- (2) For $\gamma \in \Gamma$, define $\text{Ad } \gamma \in \text{Aut}_k(A \rtimes \Gamma)$ to be $\text{Ad } \gamma(a\gamma') = \gamma a \gamma' \gamma^{-1} = \gamma(a) \text{Ad}_\Gamma \gamma(\gamma')$.
- (3) Given a weight $\lambda \in \text{Hom}_{k\text{-alg}}(A, k)$ and an A -module M , its λ -weight space is $M_\lambda := \{m \in M : am = \lambda(a)m \forall a \in A\}$.

2.1. The main definitions. We now mention the general setup under which our results are proved; for “all practical purposes”, the assumptions are simpler, and to see them, the reader should go ahead directly to §2.2.

Definition 2.2. An associative k -algebra A , together with the following data, is called a *regular triangular algebra* (denoted also by *RTA*; see [Kh2]).

- (RTA1) There exist associative unital k -subalgebras B_\pm and H of A , such that the multiplication map $: B_- \otimes_k H \otimes_k B_+ \rightarrow A$ is a vector space isomorphism (the *triangular decomposition*).
- (RTA2) There is an algebra map $\text{ad} \in \text{Hom}_{k\text{-alg}}(H, \text{End}_k(A))$, such that for all $h \in H$, $\text{ad } h$ preserves each of H, B_\pm (identifying them with their respective images in A). Moreover, $H \otimes B_\pm$ are k -subalgebras of A .

- (RTA3) There exists a free action $*$ of a group \mathcal{P} on $G := \text{Hom}_{k\text{-alg}}(H, k)$, as well as a distinguished element $0_G = 0_{\mathcal{P}} * 0_G \in G$ such that $H = H_{0_G}$ as an $\text{ad } H$ -module.
- (RTA4) There exists a subalgebra H_0 of H , and a free abelian group \mathcal{P}_0 of finite rank, such that
- (a) \mathcal{P}_0 acts freely on $G_0 := \text{Hom}_{k\text{-alg}}(H_0, k)$ (call this action $*$ as well), and
 - (b) the “restriction” map $\pi : G \rightarrow G_0$ sends $\mathcal{P} * 0_G$ onto $\mathcal{P}_0 * \pi(0_G)$, and intertwines the actions, i.e., $* \circ (\pi \times \pi) = \pi \circ *$.

For the remaining axioms, we need some notation. Fix a finite basis Δ of \mathcal{P}_0 . For each $\theta \in \mathcal{P}$ and $\theta_0 \in \mathcal{P}_0 = \mathbb{Z}\Delta$, abuse notation and define $\theta = \theta * 0_G \in G$, $\theta_0 = \theta_0 * \pi(0_G) \in G_0$. (We will differentiate between $0 \in \mathcal{P}_0$ or G_0 , and $0_G \in G$.) We call G (or $G_0, \mathcal{P}_0, \Delta$) the set of *weights* (or the *restricted weights*, *root lattice*, *simple roots* respectively).

Given $\lambda \in S \subset G$ and a module M over H (e.g., $M = (A, \text{ad})$), define the *weight space* M_λ as above, and $M_S := \bigoplus_{\lambda \in S} M_\lambda$. Given $\theta_0 \in \mathbb{Z}\Delta$, define $M_{\theta_0} := M_{\pi^{-1}(\theta_0)}$.

- (RTA5) It is possible to choose Δ , such that $B_\pm = \bigoplus_{\theta \in \mathcal{P} : \pi(\theta) \in \pm \mathbb{Z}_{\geq 0} \Delta} (B_\pm)_\theta$ (where A is an H -module via ad).
- (RTA6) $(B_\pm)_0 = (B_\pm)_{\pi^{-1}(\pi(0_G))} = k$, and $\dim_k (B_\pm)_{\theta_0} < \infty \ \forall \theta_0 \in \pm \mathbb{Z}_{\geq 0} \Delta$ (we call this *regularity*).
- (RTA7) The *property of weights* holds: for all A -modules M ,

$$\begin{aligned} A_\theta \cdot A_{\theta'} &\subset A_{\theta * \theta'} \ \forall \theta, \theta' \in \mathcal{P}, \\ A_\theta \cdot M_\lambda &\subset M_{\theta * \lambda} \ \forall \theta \in \mathcal{P}, \lambda \in G. \end{aligned}$$

- (RTA8) There exists an anti-involution i of A (i.e., $i^2|_A = \text{id}|_A$) that acts as the identity on all of H , and takes A_θ to $A_{\theta^{-1}}$ for each $\theta \in \mathcal{P}$.

Definition 2.3. An RTA is *strict* if $H = H_0, G = G_0 \supset \mathcal{P} = \mathcal{P}_0$ (whence $\pi = \text{id}|_G$).

Example. This definition is quite technical; here is our motivating example - a complex semisimple Lie algebra \mathfrak{g} . Then $A = \mathfrak{U}\mathfrak{g}$, ad is the standard adjoint action, and $H = H_0 = \text{Sym } \mathfrak{h}$, whence the set of weights is $G = G_0 = \mathfrak{h}^* \supset \mathcal{P} = \mathcal{P}_0 = \mathbb{Z}\Delta$ (the root lattice). Moreover, i is the composite of the Chevalley involution and the Hopf algebra antipode on $\mathfrak{U}\mathfrak{g}$.

Remark 2.1. We note that H is commutative by (RTA8), and (RTA6) defines augmentation ideals N_\pm of B_\pm . One change from earlier theories of the Category \mathcal{O} , is in our allowing “non-strict” RTAs in our setup; this is needed if we want to include infinitesimal Hecke algebras (not over \mathfrak{sl}_2). See [Kh2, KT] for more details.

Standing Assumption 2.1. Henceforth, A is an RTA, Γ is a finite group acting on A , and k is a field of characteristic zero if $|\Gamma| > 1$. Moreover, in $A \rtimes \Gamma$,

- (1) There is an algebra map $\text{ad} : H \rtimes \Gamma \rightarrow \text{End}_k(A)$, that restricts on H, Γ to $\text{ad} \in \text{Hom}_{k\text{-alg}}(H, \text{End}_k(A))$ and $\text{Ad} \in \text{Hom}_{\text{group}}(\Gamma, \text{Aut}_{k\text{-alg}}(A))$ respectively. Moreover, $i(\gamma(a)) = \gamma(i(a))$ for $\gamma \in \Gamma, a \in A$, and each subalgebra $R = k, N_{\pm}, H_0, H$ is preserved by $\text{ad} \xi$, for each $\xi \in H \rtimes \Gamma$. Here, the restricted $\text{ad}|_H$ and the anti-involution i are part of the RTA-structure of A .
- (2) The map $\langle \cdot, \cdot \rangle : \Gamma \times G \rightarrow G = \text{Hom}_{k\text{-alg}}(H, k)$, given by $\langle \gamma(\lambda), h \rangle := \langle \lambda, \gamma^{-1}(h) \rangle$, is an action that preserves $(\mathcal{P} * 0_G, *)$. That is, $\gamma(\theta) * \gamma(\lambda) = \gamma(\theta * \lambda)$ for all γ, θ, λ respectively in Γ, \mathcal{P}, G , and $\gamma : \mathcal{P} * 0_G \rightarrow \mathcal{P} * 0_G \forall \gamma$.

The above assumptions imply the following “compatibility”:

Lemma 2.1. *Suppose $A \rtimes \Gamma$ is a skew group ring that satisfies Assumption 2.1. Then Γ preserves 0_G , and also acts on G_0 (and \mathcal{P}_0), such that π intertwines the actions: $\forall \gamma \in \Gamma, \pi \circ \gamma = \gamma \circ \pi$ on G . Moreover, $\gamma(A_{\theta}) = A_{\gamma(\theta)} \forall \gamma \in \Gamma, \theta \in \mathcal{P}$, and $A \rtimes \Gamma$ has an anti-involution that restricts to i, i_{Γ} on A, Γ respectively.*

To show this, we need some basic results.

Lemma 2.2. *Suppose a group Γ acts on an associative unital k -algebra R by k -algebra automorphisms. Then Γ acts on $R^* : \langle \gamma(\lambda), r \rangle := \langle \lambda, \gamma^{-1}(r) \rangle$.*

- (1) *Given $\gamma \in \Gamma$, an R -module M , and a weight λ , $\gamma(M_{\lambda}) = M_{\gamma(\lambda)}$.*
- (2) *Suppose $i : R \rightarrow R$ is an anti-involution. Define $i_{\Gamma}(\gamma) = \gamma^{-1}$ for $\gamma \in \Gamma$. Then i, i_{Γ} extend to an anti-involution of $R \rtimes \Gamma$, if and only if $i(\gamma(r)) = \gamma(i(r))$ for all $r \in R, \gamma \in \Gamma$.*

Proof of Lemma 2.1. That $\gamma(A_{\theta}) = A_{\gamma(\theta)}$ follows from the above lemma. Moreover, $H = \gamma(H) = \gamma(H_{0_G}) = H_{\gamma(0_G)}$, whence γ fixes 0_G . Finally, given $\lambda \in G$, one easily checks that $\gamma(\pi(\lambda)) = \pi(\gamma(\lambda))$ on H_0 ; in turn, this implies that Γ acts on \mathcal{P}_0 (since it acts on \mathcal{P} by Assumption 2.1). \square

We used the (standard) Hopf algebra structure of $k\Gamma$ in the above results (i.e., $\Delta(\gamma) = \gamma \otimes \gamma, S(\gamma) = \gamma^{-1}, \varepsilon(\gamma) = 1$). This is further used in the following result, which also helps us rephrase (and reduce) the assumptions when $H \supset H_0$ are Hopf algebras (i.e., A is a Hopf RTA; see [Kh2]).

Lemma 2.3. *Keep the assumptions of Lemma 2.2, and suppose also that R is a Hopf algebra. We thus have Hopf algebra operations Δ, ε, S on both the (Hopf) subalgebras $R, k\Gamma$ of $R \rtimes \Gamma$.*

- (1) *These operations on $R, k\Gamma$ extend to $R \rtimes \Gamma$ (such that $R \rtimes \Gamma$ becomes a Hopf algebra), if and only if Ad is a group homomorphism from Γ to Hopf algebra automorphisms $\text{Aut}_{\text{Hopf}}(R)$.*

Now suppose that the conditions in the first part hold.

- (2) Then $\text{Ad} \in \text{Hom}_{\text{group}}(\Gamma, \text{Aut}_{\text{group}}(G))$, where $G = \text{Hom}_{k\text{-alg}}(R, k)$ is a group under convolution: $(\lambda * \mu)(r) := \sum \lambda(r_{(1)})\mu(r_{(2)})$.
- (3) If $A \supset R$ is a k -algebra containing R (with $1_A = 1_R$), so that Γ acts on A by algebra maps (with compatible restriction to R), then $\text{ad}|_R$ and $\text{Ad}|_\Gamma$ can be extended to $\text{ad} \in \text{Hom}_{k\text{-alg}}(R \rtimes \Gamma, \text{End}_k(A \rtimes \Gamma))$.

We remark that not every algebra automorphism of a Hopf algebra is a Hopf algebra automorphism; for example, if H is a non-cocommutative Hopf algebra, then the flip map $\tau : H \otimes H^{\text{cop}} \rightarrow H \otimes H^{\text{cop}}$, given by $\tau(x \otimes y) = y \otimes x$, is an algebra map but not a coalgebra map. Here, H^{cop} denotes the Hopf algebra $(H, m, \eta, \Delta^{\text{op}}, \varepsilon, S^{-1})$.

2.2. The case of Hopf algebras. Several extensively studied examples in representation theory occur with the additional data that $H \supset H_0$ are Hopf algebras (see [Kh2] for a theorem, as well as a list of examples). Thus, this is the setup one should have in mind.

The lemmas above, together with the analysis in [Kh2], show that some of the defining assumptions can be relaxed. (In particular, A is now a *Hopf RTA*, or *HRTA*, when Γ is trivial.) Let us mention the “reduced” set of axioms for A and $A \rtimes \Gamma$, obtained by combining all this.

Proposition-Definition 2.1. A *skew group ring over a Hopf RTA* is $A \rtimes \Gamma$, where all but the last part ($\Gamma(\Delta) = \Delta$) hold if and only if (see [Kh2]) A is an RTA, $H \supset H_0$ are Hopf algebras with compatible structures, and ad their usual adjoint actions.

- (1) The multiplication map: $B_- \otimes_k H \otimes_k B_+ \rightarrow A$ is a vector space isomorphism. Here H, B_\pm are associative unital k -subalgebras of A . Moreover, the “Cartan part” H is a commutative Hopf algebra.
- (2) H contains a sub-Hopf algebra H_0 (with groups of weights G, G_0 respectively), and G_0 contains a free abelian group of finite rank $\mathcal{P}_0 = \mathbb{Z}\Delta$. Here, Δ is a basis of \mathcal{P}_0 , chosen such that

$$B_\pm = \bigoplus_{\theta \in G: \pi(\theta) \in \pm \mathbb{Z}_{\geq 0} \Delta} (B_\pm)_\theta = \bigoplus_{\theta_0 \in \pm \mathbb{Z}_{\geq 0} \Delta} (B_\pm)_{\theta_0}$$

(the summands are weight spaces under the usual adjoint actions). Each summand in the second sum is finite-dimensional, and $(B_\pm)_{0_G} = (B_\pm)_0 = k$.

- (3) There exists an anti-involution i of A , such that $i|_H = \text{id}|_H$.
- (4) Γ is a finite group, that acts on B_\pm, H, H_0 (and hence on A), such that the action of each $\gamma \in \Gamma$
- on B_\pm is by algebra automorphisms,
 - on H (and hence on H_0) is by Hopf algebra automorphisms,
 - on A commutes with the anti-involution i , and
 - induced on G_0 preserves Δ .

Remark 2.2. First, we do not assume here, that H is cocommutative; nevertheless, $H \otimes B_\pm \cong B_\pm \rtimes H$, the smash product algebras. Next, that Γ

preserves Δ is included, because it will be needed later to show that every Verma module has a unique simple quotient; see Proposition 6.2 below.

Recently, Bazlov and Berenstein defined a class of algebras that encompasses symmetrizable Kac-Moody Lie algebras and their quantum groups, and rational Cherednik algebras. We now mention their connection to skew group rings (so one may now try to apply their results in this setup).

Definition 2.4. A k -algebra A has *triangular decomposition over a bialgebra* H' , if A has distinguished subalgebras H', U^\pm such that

- H' acts covariantly from the left on U^- , and from the right on U^+ ;
- the multiplication map $: U^- \otimes H' \otimes U^+ \rightarrow A$ is a vector space isomorphism, that makes $U^- \otimes H'$ and $H' \otimes U^+$ isomorphic to the smash products $U^- \rtimes H'$ and $H' \ltimes U^+$ (by the above actions) respectively;
- there exist H' -equivariant (via the counit ε) characters $\epsilon^\pm : U^\pm \rightarrow k$.

Proposition 2.1. *Skew group rings over HRTAs are examples of algebras with triangular decomposition over $H \rtimes \Gamma$.*

Proof. Set $U^\pm = B_\pm$ and $H' = H \rtimes \Gamma$. Moreover, define the two actions of $H \rtimes \Gamma$ (on all of A , in fact) to be: $h \triangleright a := \text{ad } h(a)$, $a \triangleleft h := \text{ad } S(h)(a)$. Since B_\pm are direct sums of $\text{ad } H$ -weight spaces, and closed under $\text{ad } \Gamma$, hence one can check that these are valid left and right $H \rtimes \Gamma$ -actions (note that $S^2 = \text{id}|_{H'}$, since H is commutative and Γ is cocommutative). It is now easy to verify the first two conditions. Finally, define the characters ϵ^\pm to have augmentations N_\pm . Over here, since H is a Hopf algebra, we have $0 = \varepsilon : H \rightarrow k$ (and ε can be extended to $H \rtimes \Gamma$). It is now easy to verify that ϵ^\pm are H' -equivariant (via ε), for we verify on each θ -weight space, that $\epsilon^\pm(\text{ad}(h\gamma)(u_\theta^\pm)) = \delta_{\gamma(\theta), \varepsilon} \varepsilon(h) u_{\gamma(\theta)}^\pm = \varepsilon(h\gamma) \epsilon^\pm(u_\theta^\pm)$. \square

2.3. Examples.

- (1) The degenerate example is that of an RTA A , where we take $\Gamma = 1$.
- (2) The *wreath product* $S_n \wr A$ is defined to be $A^{\otimes n} \rtimes S_n = S_n \wr A$, where S_n is the group of permutations of $\{1, \dots, n\}$, and A is a (strict) (Hopf) RTA. By [Kh2], $A^{\otimes n}$ is also a (strict) (Hopf) RTA, with simple roots $\Delta = \coprod_i \Delta_i$, and weights $G = G_A^n, \mathcal{P} = \mathcal{P}_A^n$, and so on.

Define $f_i : A \hookrightarrow A^{\otimes n} \subset S_n \wr A$, sending a to the product of $1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)} \otimes 1_\Gamma$; then the relations are $s_{ij} f_i(a) = f_j(a) s_{ij}$, where $a \in A$, and s_{ij} is the transposition that exchanges i and j . Thus, $\sigma(\mathcal{P}_i) = \mathcal{P}_{\sigma(i)}$, and $\sigma(\theta * \lambda) = \sigma(\theta) * \sigma(\lambda) \forall \theta \in \mathcal{P}_A^n, \lambda \in G_A^n$.

Moreover, $\sigma(\alpha_i) = \alpha_{\sigma(i)}$, where $\sigma \in S_n$, $\alpha \in \Delta$ is a simple root, and $\alpha_i := f_i^*(\alpha) : (H_0)_i \rightarrow k$. (So $\sigma(\Delta_i) = \Delta_{\sigma(i)}$.) Finally, define

$$\text{ad}(h_1 \otimes \dots \otimes h_n \otimes \sigma)(a_1 \otimes \dots \otimes a_n) := \otimes_j \text{ad } h_j(a_{\sigma^{-1}(j)}).$$

One checks that each $\text{Ad } \sigma$ acts by a Hopf algebra automorphism if H (and hence $H^{\otimes n}$) is a Hopf algebra. Thus, $S_n \wr A$ satisfies all the standing assumptions.

- (3) If $A \rtimes \Gamma$ is a skew group ring over a (strict) (Hopf) RTA, then so is $A \rtimes \Gamma'$, for any subgroup Γ' of Γ .
- (4) For any finite Γ , $A \otimes k\Gamma$ is a skew group ring if $A^\Gamma = A$.
- (5) If $A_i \rtimes \Gamma_i$ are skew group rings that satisfy the above assumptions, then we know by [Kh2], that $A = \otimes_i A_i$ is an RTA, the set G of weights is $\times_i G_i$, and $\Gamma = \times_i \Gamma_i$ acts on A , via:

$$(\gamma_1, \dots, \gamma_n) \cdot (a_1 \otimes \dots \otimes a_n) = (\gamma_1(a_1) \otimes \dots \otimes \gamma_n(a_n)).$$

One can check that $A \rtimes \Gamma$ also satisfies all the assumptions above.

Finally, if H_i is a Hopf algebra for all i , then so is $H = \otimes_i H_i$. If each $\text{Ad } \gamma_i$ acts as a Hopf algebra automorphism on A_i , then the same property holds for $\text{Ad } \Gamma$ acting on A .

3. THE BERNSTEIN-GELFAND-GELFAND CATEGORY

We now introduce the main object of study in this article.

Definition 3.1. The *BGG category* \mathcal{O} is the full subcategory of finitely generated H -semisimple $A \rtimes \Gamma$ -modules, with finite-dimensional weight spaces and a locally finite action of B_+ , i.e., $\forall m \in M \in \mathcal{O}$, $\dim(B_+ m) < \infty$.

Then \mathcal{O} is closed under quotienting, and every object M in \mathcal{O} has a locally finite action of $(H \otimes B_+ \otimes k\Gamma)$. Moreover, viewing each $M \in \mathcal{O}$ as merely an A -module, we can show the following lemma, since Γ is finite.

Lemma 3.1. *Suppose \mathcal{O}_A is the Category \mathcal{O} for $|\Gamma| = 1$. Then $\mathcal{O} = \mathcal{O}_{A \rtimes \Gamma}$ equals $A \rtimes \Gamma - \text{Mod} \cap \mathcal{O}_A := \{M \in A \rtimes \Gamma - \text{Mod} : \text{Res}_A^{A \rtimes \Gamma} M \in \mathcal{O}_A\}$.*

Let us first show that complete reducibility for finite-dimensional A -modules implies it for $A \rtimes \Gamma$ -modules; the special case $A = (\mathfrak{U}\mathfrak{g})^{\otimes n}$, $\Gamma = S_n$ was stated in Proposition 1.1 above (but one can also state that result for $U_q(\mathfrak{g})$, for instance). We use the following general homological result.

Proposition 3.1. *Given a finite group Γ acting on an algebra R over a field of characteristic zero, suppose $\mathcal{P} \subset R - \text{Mod}$ and $\mathcal{D} \subset (R \rtimes \Gamma) - \text{Mod}$ are full abelian subcategories of finite-dimensional modules, with each $D \in \mathcal{D}$ satisfying: $\text{Res}_R^{R \rtimes \Gamma} D \in \mathcal{P}$. If \mathcal{P} is a semisimple category, then so is \mathcal{D} .*

The second part of Proposition 1.1 now follows, by setting \mathcal{P}, \mathcal{D} to be the categories of finite dimensional $A, A \rtimes \Gamma$ -modules respectively. The first part will be shown after Remark 4.2.

Proof. It suffices to show that $\text{Hom}_{\mathcal{D}}(M, -)$ is exact for each object M of \mathcal{D} (since \mathcal{D} is abelian, the long exact sequence of $\text{Ext}_{\mathcal{D}}$'s vanishes). Since \mathcal{D} is also full, we use [Mac, Equation (A.1), Appendix], and compute:

$$\text{Hom}_{\mathcal{D}}(M, -) = \text{Hom}_{R \rtimes \Gamma}(M, -) = \left(\text{Hom}_R(\text{Res}_R^{R \rtimes \Gamma} M, \text{Res}_R^{R \rtimes \Gamma} -) \right)^\Gamma$$

By Maschke's Theorem (over characteristic zero), taking Γ -invariants is an exact functor (since everything is finite-dimensional here), as is $\text{Res}_R^{R \rtimes \Gamma} -$:

$\mathcal{D} \rightarrow \mathcal{P}$. By semisimplicity in the full subcategory \mathcal{P} , $\text{Hom}_{\mathcal{P}}(\text{Res}_{\mathbb{R}}^{\mathbb{R} \rtimes \Gamma} M, -)$ is also exact. Thus their composite is exact as well. \square

4. SUMMARY OF RESULTS

We now summarize our main results.

Standing Assumption 4.1. Suppose $A_i \rtimes \Gamma_i$ are skew group rings over RTAs for $1 \leq i \leq n$, each of which satisfies Standing Assumption 2.1 above. Suppose also that each Γ_i preserves $\Delta_i \subset (G_0)_i$, and that k is algebraically closed if any Γ_i is nontrivial.

Then all this also holds for $A \rtimes \Gamma$, where $A := \otimes_i A_i$ and $\Gamma := \times_i \Gamma_i$. Hence, we will state results only for $A \rtimes \Gamma$ wherever possible, because it combines all the functorial constructions above (see (1.1)). One can thus use A as an RTA or as $\otimes_i A_i$. To see the results for any of the “subcases”, take $n = 1$ (to study only Clifford theory), Γ to be trivial (to study only RTAs), etc.

Let \mathcal{C} be the category of finite-dimensional H -semisimple $H \rtimes \Gamma$ -modules, and X the isomorphism classes of simple objects in \mathcal{C} (so $X = G$ if $|\Gamma| = 1$). Then

- \mathcal{C} is semisimple.
- X classifies the simple objects in $\mathcal{O} = \mathcal{O}_{A \rtimes \Gamma}$, and for each $x \in X$, there is a *Verma module* $Z(x) \in \mathcal{O}$ with unique simple quotient $V(x)$.
- Given x , there is a Γ -orbit $\lambda_x \in G/\Gamma$ (its “set of weights”), such that
 - for each $\lambda \in G$, there exists at least one - and only finitely many - $x \in X$ such that $\lambda \in \lambda_x$.
 - (“Weyl Character Formula 1”) $V(x) \cong \bigoplus_{\lambda \in \lambda_x} V_A(\lambda)^{\oplus \dim x_\lambda}$ as A -modules in \mathcal{O}_A , where $V_A(\lambda)$ is simple in \mathcal{O}_A .
 - The center $Z := \mathfrak{Z}(A \rtimes \Gamma)$ acts on a Verma or simple module by a *central character* $\chi_x : Z \rightarrow k$.

Next, we define duality functors F on the *Harish-Chandra categories* \mathcal{H} containing \mathcal{O} and \mathcal{C} , using the anti-involution i on A and H respectively.

- F is exact, contravariant, and involutive on \mathcal{C} and \mathcal{H} .
- $F(V(x)) = V(F(x))$.
- Simple modules $V(x), V(x')$ have non-split extensions in \mathcal{O} if and only if they or $V(F(x')), V(F(x))$ are the first two Jordan-Holder factors in a composition series for a Verma module.

For example, if $|\Gamma| = 1$, then $X = G$ and $F(V(\lambda)) \cong V(\lambda) \forall \lambda \in G$.

Definition 4.1. Fix $x \in X$.

- (1) Define $CC(x) = S^4(x)$ to be $\{x' \in X : \chi_{x'} = \chi_x\}$.
- (2) Define $S^3(x)$ to be the symmetric and transitive (or equivalence) closure of $\{x\}$ in X , under the relations (a) $x \rightarrow x'$ if $V(x)$ is a subquotient of $Z(x')$, and (b) $x \rightarrow_F x'$ if $F(x) = x'$.
- (3) Define $S'(x)$ to be the equivalence closure of $\{x\}$ in X under only the relation (a) above.

- (4) Define the following partial order on X : $x \leq x'$ if $x = x'$, or there exist $\lambda \in \lambda_x, \lambda' \in \lambda_{x'}$, with $\pi(\lambda') \in (\mathbb{Z}_{\geq 0}) * \pi(\lambda)$.
- (5) Given $S \subset X \ni x$, define $S^{\leq x} := \{s \in S : s \leq x\}$ (and as a special case, $S^{\leq \lambda}$ if $S \subset G$ and $|\Gamma| = 1$). Similarly define $S^{< x}$ and $S^{< \lambda}$.
- (6) $S^2(x) := \{\pi(\lambda_{x'}) : x' \in S^3(x)\}$, $S^1(x) := \{\pi(\lambda_{x'}) : x' \in S^3(x)^{\leq x}\}$.
- (7) $A \rtimes \Gamma$ satisfies *Condition* (S1), (S2), (S3), or (S4) if the corresponding sets $S^n(x)$ are finite for all $x \in X$.
- (8) The *block* $\mathcal{O}(x)$ is the full subcategory of \mathcal{O} , comprising all objects in \mathcal{O} , each of whose simple subquotients is in $\{V(x') : x' \in S^3(x)\}$.
- (9) Given skew group rings $A_i \rtimes \Gamma_i$ over RTAs for $1 \leq i \leq n$, each of which satisfies Standing Assumption 4.1 above, define X_i similar to X , but over $H_i \rtimes \Gamma_i$ for all i . Given $\lambda_i \in G_i \forall i$, the *simple objects* over $\lambda = (\lambda_1, \dots, \lambda_n)$ in the four setups in (1.1) are, respectively,

$$\lambda_i, \{x_i \in X_i : \lambda_i \in \lambda_{x_i}\}, \lambda, \{\mathbf{x} \in X : \lambda \in \lambda_{\mathbf{x}}\}. \quad (4.1)$$

Remark 4.1.

- (1) As we see below, $X = \times_i X_i$ here, so $\lambda_{\mathbf{x}} \in \times_i G_i = G$, as it should.
- (2) That Verma modules $Z(x)$ are indecomposable and have a unique simple quotient $V(x)$, is related to the fact that $Z(x)$ is the projective cover of $V(x)$ in a “truncated” subcategory of \mathcal{O} . (See Lemma 12.1.)
- (3) The relation between simple objects in the four setups for Category \mathcal{C} (and \mathcal{O}) is expected from diagram (1.2), and “indicated” in (4.1). Its analogue in \mathcal{O} forms the “front face” of the “cube of simple objects” below. We show later, that $V(\mathbf{x}) := \otimes_i V_i(x_i) \in \mathcal{O}_{A \rtimes \Gamma}$. We now state all this using diagrams; in them, “ \rtimes ” really is some sort of induction, or an “inverse” to “restriction” (from $A_i \rtimes \Gamma_i$ to A_i).
- (4) The Conditions (S) are not uncommon: (S3) holds for complex semisimple Lie algebras, (nontrivially deformed) infinitesimal Hecke algebras, and both their quantum analogues (e.g., see the example after Remark 4.2, [Kh1, Kh2, GK] respectively).

Theorem 4.1. *Fix $1 \leq m \leq 4$, and work in the above setup.*

- (1) $X = \times_i X_i$.
- (2) Inside any such $A \rtimes \Gamma$, $\gamma(S_A^m(\lambda)) = S_A^m(\gamma(\lambda)) \forall \lambda \in G, \gamma \in \Gamma$.
- (3) Given RTAs A_i and $\lambda_i \in G_i$, $S_A^m(\lambda_1, \dots, \lambda_n) = \times_i S_{A_i}^m(\lambda_i)$.

Proposition 4.1.

- (1) *The following diagram of “posets of simple objects in \mathcal{O} ” commutes (in the spirit of diagram (1.2)) in between various \mathcal{C} ’s:*

$$\begin{array}{ccc} \{G_i\} & \xrightarrow{\rtimes} & \{X_i\} \\ \times \downarrow & & \times \downarrow \\ G = \times_i G_i & \xrightarrow{\rtimes} & X = \times_i X_i \end{array}$$

- (2) Moreover, given $\lambda_i \in G_i$ and $x_i \in X_i$ (for all i) such that $\lambda_i \in \lambda_{x_i}$, the three constructions of \otimes_i , F , and “ \rtimes ” form a commuting “cube of simple objects” in various \mathcal{O} ’s:

$$\begin{array}{ccccc}
 & & \{V_i(\lambda_i)\} & \xrightarrow{\rtimes} & \{V_i(F(x_i))\} \\
 & \nearrow F & \downarrow \rtimes & & \nearrow F \\
 \{V_i(\lambda_i)\} & \xrightarrow{\otimes} & \{V_i(x_i)\} & & \downarrow \otimes \\
 \downarrow \otimes & & \downarrow \rtimes & & \downarrow \otimes \\
 V(\lambda) & \xrightarrow{\rtimes} & V(F(\mathbf{x})) & & \\
 \downarrow \otimes & \nearrow F & \downarrow \rtimes & \nearrow F & \\
 V(\lambda) & \xrightarrow{\rtimes} & V(\mathbf{x}) & &
 \end{array}$$

- (3) The “corresponding” cube in between various \mathcal{C} ’s commutes.
 (4) For all $\gamma \in \Gamma$, $\lambda_i \in G_i$, and $x_i \in X_i$ with $\lambda_i \in \lambda_{x_i} \forall i$, the “Verma cube” below, commutes (here, $Z(\mathbf{x}) := \otimes_i Z_i(x_i)$):

$$\begin{array}{ccccc}
 & & \{Z_i(\gamma(\lambda_i))\} & \xrightarrow{\rtimes} & \{Z_i(x_i)\} \\
 & \nearrow \gamma(\cdot) & \downarrow \rtimes & & \nearrow \gamma(\cdot)=\text{id} \\
 \{Z_i(\lambda_i)\} & \xrightarrow{\otimes} & \{Z_i(x_i)\} & & \downarrow \otimes \\
 \downarrow \otimes & & \downarrow \rtimes & & \downarrow \otimes \\
 Z(\lambda) & \xrightarrow{\rtimes} & Z(\mathbf{x}) & & \\
 \downarrow \otimes & \nearrow \gamma(\cdot) & \downarrow \rtimes & \nearrow \gamma(\cdot)=\text{id} & \\
 Z(\lambda) & \xrightarrow{\rtimes} & Z(\mathbf{x}) & &
 \end{array} \tag{4.2}$$

We also get a commuting cube by replacing all Z ’s by V ’s in part (4). Note also that the poset structure on each set in part (1) is needed to prove that the \mathcal{O} ’s are highest weight categories.

We now state our main theorems; the rest of this article is devoted to proving them. Note that \mathcal{O} is taken to mean the BGG Category in any of the four setups. Finally, for an explicit statement of parts 1(b) and 1(c) below, see Proposition 17.1.

Theorem 4.2. *Suppose the standing assumptions 4.1 above hold.*

- (1) *Each of the following conditions holds in one setup (see (1.1)), if and only if it holds in any of the other three:*
- (a) *Finite-dimensional modules in \mathcal{O} are completely reducible.*
 - (b) *For a fixed $\lambda = (\lambda_1, \dots, \lambda_n) \in \times_i G_i$, $V(x)$ is finite-dimensional for any simple object x over λ .*

- (c) For a fixed $\lambda = (\lambda_1, \dots, \lambda_n) \in \times_i G_i$, $Z(x)$ has finite length for any simple object x over λ .
- (d) \mathcal{O} (equivalently, every Verma module $Z(x)$) is finite length.
- (e) Any of Conditions (S1)-(S4) holds (for a fixed $1 \leq n \leq 4$); for (S4), we also need that $\mathfrak{Z}(A \rtimes \Gamma) = \mathfrak{Z}(A)^\Gamma \subset A$.
- (2) We have the following sequence of implications of the Conditions (S): (S3) \Rightarrow (S2) \Rightarrow (S1); moreover, (S4) \Rightarrow (S3) if $\mathfrak{Z}(A \rtimes \Gamma) \subset A$.

Theorem 4.3. *Suppose the standing assumptions 4.1 above hold.*

- (1) *If $A \rtimes \Gamma$ satisfies Condition (S1), then \mathcal{O} is finite length, and hence splits into a direct sum of abelian, finite length blocks $\mathcal{O}(x)$.*
- (2) *If $A \rtimes \Gamma$ satisfies Condition (S2), then each block has enough projectives, each with a filtration whose subquotients are Verma modules.*
- (3) *If $A \rtimes \Gamma$ satisfies Condition (S3), then each block $\mathcal{O}(x)$ is a highest weight category, equivalent to the category $(\text{Mod } -B)^{fg}$ of finitely generated right modules over a finite-dimensional k -algebra $B = B_x$.*

In the last part, moreover, many different notions of block decomposition all coincide, and (a modified form of) BGG Reciprocity (see [BGG]) holds in \mathcal{O} with a symmetric (modified) Cartan matrix.

Remark 4.2. One can add other (equivalent) setups, for example $A \rtimes \Gamma'$, where Γ' is any finite group that acts “nicely” on A, B_\pm, G, Δ .

Example: Suppose $\Gamma = S_n$ and $A = (\mathfrak{U}\mathfrak{g})^{\otimes n}$ for a complex semisimple Lie algebra \mathfrak{g} . Then the “dominant integral” $x \in X$ are precisely the simple objects over the dominant integral weights $(P_{\mathfrak{g}}^+)^n$ of $\mathfrak{g}^{\oplus n}$. Every finite-dimensional module is in $\mathcal{O}_{A \rtimes \Gamma}$, and is completely reducible.

Theorem A.5 in [Mac] explicitly describes each $x \in X$ (the construction for $V(x)$ is similar): choose a partition $n_1 + \dots + n_l = n$, and for each j , pick $\lambda_j \in \mathfrak{h}^*$ pairwise distinct, and simple S_{n_j} -modules N_j . Then $x = \text{Ind } \otimes_{j=1}^l [(\mathbb{C}^{\lambda_j})^{\otimes n_j} \otimes_{\mathbb{C}} N_j]$, where one induces from $\otimes_{j=1}^l (S_{n_j} \wr \mathfrak{U}\mathfrak{h})$ to $S_n \wr \mathfrak{U}\mathfrak{h}$.

For example, fix any $\lambda \in \mathfrak{h}^*$. The module corresponding to the partition $n = n_1$ and the one-dimensional simple S_n -module, is $V(x) = V(\lambda)^{\otimes n}$ (and S_n acts by permuting the factors). (This proves the first part of Proposition 1.1, since the modules $V(\lambda)^{\otimes n}$ have unequal formal characters for pairwise distinct λ .) More generally, $V(\lambda)^{\otimes n} \otimes_{\mathbb{C}} E$ is simple in \mathcal{O} for any simple S_n -module E .

Finally, all the Conditions (S) hold for $\mathfrak{U}\mathfrak{g}$ (and hence for $A \rtimes S_n$, by above results). This is because by the theory of central characters and Harish-Chandra’s theorem, $S^4(\lambda) = W_{\mathfrak{g}} \bullet \lambda$ (so the strict HRTA $\mathfrak{U}\mathfrak{g}$ satisfies Condition (S4)), where $W_{\mathfrak{g}}$ is the (finite) Weyl group of \mathfrak{g} , and \bullet its twisted action. This also gives the usual and twisted actions of $S_n \wr W_{\mathfrak{g}}$ on the set of weights $(\mathfrak{h}_{\mathfrak{g}}^*)^{\oplus n}$. The twisted action is “good”, since

$$(\sigma \mathbf{w}) \bullet \lambda = (\sigma(\mathbf{w})\sigma) \bullet \lambda = \sigma(\mathbf{w} \bullet \lambda) = \sigma(\mathbf{w}) \bullet \sigma(\lambda)$$

for all $\lambda \in (\mathfrak{h}_{\mathfrak{g}}^*)^{\oplus n}$, $\mathbf{w} \in W_{\mathfrak{g}}^n$, and $\sigma \in S_n$. We now adapt the above results here; note that $R_{\mathfrak{g}} = S_n \wr \mathfrak{U}\mathfrak{g}$.

Proposition 4.2.

- (1) $R_{\mathfrak{g}}$ is of finite representation type.
- (2) If \mathcal{O}' is the category of $M \in R_{\mathfrak{g}} - \text{Mod}$ such that $\text{Res}_A^{R_{\mathfrak{g}}} M \in \mathcal{O}_A$, then \mathcal{O}' is a direct sum of subcategories $\mathcal{O}'(\lambda)$. For a given summand, the modules in it are of finite length, and all their simple subquotients have highest weights only in $S_n(W_{\mathfrak{g}}^n \bullet \lambda)$ for various $\lambda \in \Theta := (\mathfrak{h}_{\mathfrak{g}}^*)^{\oplus n} / (S_n \wr W_{\mathfrak{g}}, \bullet)$.
- (3) Θ is precisely the set of central characters of $R_{\mathfrak{g}}$, where $R_{\mathfrak{g}}$ has center $(\mathbb{Z}(\mathfrak{U}\mathfrak{g})^{\otimes n})^{S_n}$.

Proposition 4.1 and all the theorems in this section are proved in §17 below, and Proposition 4.2 is shown in Section 11.

5. THE FIRST SETUP - SKEW GROUP RINGS

We start by relating \mathcal{O}_A and $\mathcal{O}_{A \rtimes \Gamma}$. The first thing to do is to identify a set which will characterize the simple objects in $\mathcal{O}_{A \rtimes \Gamma}$.

Definition 5.1.

- (1) Denote by Γ_1 the set of singly generated Γ -modules.
- (2) Let \mathcal{C} denote the abelian category of finite-dimensional H -semisimple $(H \rtimes \Gamma)$ -modules.
- (3) Let Y denote the set of isomorphism classes of objects in \mathcal{C} generated by a weight vector v_{λ} (for some $\lambda \in G$).
- (4) Let $X \subset Y$ denote the isomorphism classes of simple objects in \mathcal{C} .

If $m \in M$ is a weight vector in $M \in X$ simple, then

$$M = (H \rtimes \Gamma)m = k\Gamma(k \cdot m)$$

so that every $M \in X$ is of the form $M = k\Gamma/I$ for some ideal I (as Γ -modules). Now suppose $M \in Y$; thus, M is of the form $k\Gamma/I$. Note that to fix an $(H \rtimes \Gamma)$ -structure on the module involves fixing the weight vector $v_{\lambda} \in k\Gamma/I$. This is equivalent to choosing which ideal J of $k\Gamma$ to quotient out by, so that $k\Gamma/J \cong k\Gamma/I \in \Gamma_1$ (as Γ -modules). Hence,

$$Y = \{(\lambda, M, [J]) : \lambda \in G, M \in \Gamma_1, k\Gamma/J \cong M\}, \quad (5.1)$$

where J is assumed to be an ideal of $k\Gamma$, that annihilates v_{λ} . (We thus map $1 \mapsto v_{\lambda}$, and kill $h - \lambda(h) \cdot 1$ for all $h \in H$.) Over here, we only take isomorphism classes of ideals $[J]$. For example, if $v_{\lambda} \in M_{\lambda}$ generates M , then so does $\gamma v_{\lambda} \in M_{\gamma(\lambda)}$.

Definition 5.2. λ_y (for $y \in Y$) is defined to be this λ ; or more precisely, λ_y is the Γ -orbit $[\lambda] \in G/\Gamma$. (However, we will abuse notation and say $\lambda = \lambda_y$ - or $\lambda = \lambda_x$, if $y = x \in X$ - instead of $\lambda \in \lambda_y$.)

Remark 5.1. Note that given $M \in \Gamma_1$, not all $\lambda \in G$ occur in the set of triples in Y (as the first coordinate). For example, if M is the trivial representation of Γ , then $\gamma - 1$ annihilates M for all $\gamma \in \Gamma$, so the only permissible weights here are $\lambda \in G^\Gamma$.

However, for all $\lambda \in G$, there exists $y \in Y$ such that $\lambda = \lambda_y$. For we take the one-dimensional H -module $k^\lambda = k \cdot v_\lambda$, where $h v_\lambda = \lambda(h) v_\lambda$ for all $h \in H$. We now induce this, to get $M = \text{Ind}_H^{H \rtimes \Gamma} k^\lambda$. Thus $M \cong k\Gamma$ as k -vector spaces, and $(\lambda, M, 0) \in Y$.

Now consider the special case $\Gamma = 1$; all objects in X are one-dimensional, and $X = Y = G = \text{Hom}_{k\text{-alg}}(H, k)$. These are the maximal vectors, from which one induces Verma modules. Moreover, all objects in \mathcal{C} are H -semisimple. This last part is true in general:

Proposition 5.1. *Every object of \mathcal{C} is completely reducible.*

Proof. Use Proposition 3.1 with $R = H$, \mathcal{P} the category of finite-dimensional H -semisimple H -modules, and $\mathcal{D} = \mathcal{C}$ as above. \square

The following is the other main result of this section.

Theorem 5.1. *For each $\lambda \in G$, there exists one - and if k is algebraically closed, only finitely many - $x \in X$ with $\lambda = \lambda_x$.*

Proof. We first show that the set is nonempty. Choose $y \in Y$ such that $\lambda = \lambda_y$ (by Remark 5.1), and look at a composition series $0 \subset E_1 \subset \dots$ (as above) for $E = M_y$, as $(H \rtimes \Gamma)$ -modules. Then all subquotients are H -semisimple, so by character theory, $(E_{i+1}/E_i)_\lambda \neq 0$ for some i . Then $\lambda = \lambda_x$, where $x = E_{i+1}/E_i \in X$.

If Γ is trivial, then so is the second part of the result. Now suppose that $|\Gamma| > 1$ and k is algebraically closed of characteristic zero. To show that this set is finite, use [Mac, Theorem A.5], setting $R = H$. Now, the set of subgroups of Γ is finite; hence so is the set $\{(\Gamma', M)\}$, where Γ' is a subgroup of Γ , and M is a simple Γ' -module (up to isomorphism).

The theorem now says that every simple $H \rtimes \Gamma$ -module is of the form $\text{Ind}_{H \rtimes \Gamma'}^{H \rtimes \Gamma}(k^\lambda \otimes (\Gamma')^\mu)$ for some simple H -module k^λ and simple Γ' -module $(\Gamma')^\mu$, where Γ' fixes λ . (Note that since we are only concerned here with finite-dimensional representations, a simple H -module is one-dimensional by Lie's theorem; we denote it by k^λ as above.) The structure is given here by

$$(h \otimes \gamma)(v_\lambda \otimes w_\mu) = h\gamma \cdot v_\lambda \otimes \gamma \cdot w_\mu = \lambda(h) \cdot \gamma v_\lambda \otimes \gamma w_\mu \quad \forall h \in H, \forall \gamma \in \Gamma'$$

(where we fix a group action $: \Gamma' \rightarrow \text{Aut}_k(k^\lambda)$). Fixing λ and γ , the set of (Γ', μ) 's is finite, hence we are done. \square

6. VERMA AND STANDARD MODULES

Unless otherwise specified, the functor Ind denotes $\text{Ind}_{(H \otimes B_+) \rtimes \Gamma}^{A \rtimes \Gamma}$ henceforth. Given a finite-dimensional $(H \otimes B_+) \rtimes \Gamma$ -module E , we can define the induced module $\text{Ind } E \in \mathcal{O}$. Given a finite-dimensional $(H \rtimes \Gamma)$ -module

E , we also define the (*universal*) *standard module* $\Delta(E)$ as follows: we first give E an $(H \otimes B_+) \rtimes \Gamma$ -module structure, by

$$(h \otimes n_+ \otimes \gamma)e = 0, \quad (h \otimes 1 \otimes \gamma)e = h\gamma e$$

for each $h \in H$, $\gamma \in \Gamma$, $e \in E$, and $n_+ \in N_+$. Now define the induced module $\Delta(E)$, to be $\Delta(E) = \text{Ind } E$. The following properties are standard.

Proposition 6.1. *Suppose E is a finite-dimensional $(H \otimes B_+) \rtimes \Gamma$ -module.*

- (1) $\text{Ind } E \cong B_- \otimes_k E$, as free B_- -modules.
- (2) If E has a basis of weight vectors, then $\text{ch}_{\text{Ind } E} = \text{ch}_{B_-} \text{ch}_E$.
- (3) If $E' \subset E$ is an $(H \otimes B_+) \rtimes \Gamma$ -submodule and E is H -semisimple (as above), then $\text{Ind}(E/E') \cong \text{Ind } E / \text{Ind } E'$.

We now recall a few concepts from [Kh2]:

- (1) G_0 has a partial ordering (via the base of simple roots Δ): $\mu \leq \lambda$ if and only if there exists $\theta_0 \in \mathbb{Z}_{\geq 0}\Delta$ such that $\theta_0 * \mu = \lambda$. This induces a partial order on G : $\lambda \geq \mu$ if $\lambda = \mu$ in G , or $\pi(\lambda) > \pi(\mu)$ in G_0 .
- (2) The *formal character* of $M \in \mathcal{O}$ is $\text{ch}_M := \sum_{\lambda \in G} (\dim_k M_\lambda) e(\lambda)$.

(Note that Proposition 6.1 used the latter notion.) We now relate the Γ -action to the partial ordering.

Standing Assumption 6.1. Henceforth, Γ acts by order-preserving transformations on G_0 . In other words, $\mu \leq \lambda \Rightarrow \gamma(\mu) \leq \gamma(\lambda) \ \forall \gamma \in \Gamma$.

Remark 6.1. For instance, if $\Gamma = 1$, or $\Gamma = S_n$ and $A \rtimes \Gamma$ is the wreath product $S_n \wr A$, then this assumption holds. It also clearly holds when $A \rtimes \Gamma$ is built up from subalgebras $A_i \rtimes \Gamma_i$ (as discussed above), and each Γ_i preserves $\Delta_i \subset (G_i)_0$. Moreover, this assumption is reasonable, as the subsequent lemma shows.

Lemma 6.1. *Suppose a set G_0 contains a free abelian group (denoted by $\mathbb{Z}\Delta_0 = \bigoplus_{\alpha \in \Delta_0} \mathbb{Z}\alpha$) with a free action $*$ on G_0 , that restricts to addition on $\mathbb{Z}\Delta_0$. Define a partial order on G_0 by: $\lambda \geq \mu$ if and only if $\lambda \in (\mathbb{Z}_{\geq 0}\Delta_0) * \mu$. Suppose also that a group Γ_0 acts on G_0 , preserving $\mathbb{Z}\Delta_0$ and the action $*$.*

- (1) *The following are equivalent, for a given $\gamma \in \Gamma_0$:*
 - (a) $\gamma^{\pm 1}(\alpha) \in \Delta_0$ for all $\alpha \in \Delta_0$.
 - (b) For each $\alpha \in \Delta_0$, there exists $n_\alpha \in \mathbb{N}$, so that $\gamma^{\pm 1}(n_\alpha \alpha) > 0$.
 - (c) If $\alpha \in \Delta_0$ then $\gamma^{\pm 1}(\alpha) > 0$.
 - (d) If $\lambda \geq 0$ then $\gamma^{\pm 1}(\lambda) \geq 0$.
 - (e) γ, γ^{-1} act on G_0 by order-preserving automorphisms.
- (2) *If the conditions in the first part are satisfied, and γ has finite order in Γ_0 , then $\gamma(\lambda) \not\leq \lambda$ for all $\lambda \in G_0$.*
- (3) *Under the assumptions of Standing Assumption 2.1, if for all $\alpha \in \Delta$, there exists $n_\alpha \in \mathbb{N}$ with $(B_+)_{n_\alpha \alpha} \neq 0$, then the first part holds for $G_0 = G, \Gamma_0 = \Gamma, \Delta_0 = \Delta$.*

Note that we do not need Δ_0 or Γ_0 to be finite in this result.

Proof.

- (1) The cyclic chain of implications is easy to prove.
- (2) If $\gamma(\lambda) < \lambda$ for some λ, γ , then $\gamma^{i+1}(\lambda) < \gamma^i(\lambda) \forall i$, by the previous part. Hence if $\gamma^n = 1$ in Γ , then we get a contradiction:

$$\lambda = \gamma^n(\lambda) < \gamma^{n-1}(\lambda) < \cdots < \gamma(\lambda) < \lambda.$$

- (3) For $\gamma \in \Gamma, \lambda \in G$, $\gamma : A_\lambda \rightarrow A_{\gamma(\lambda)}$ by Lemma 2.1 (since A is an ad H -module). In particular, $0 \neq \gamma((B_+)_{n_\alpha \alpha}) \subset A_{\gamma(n_\alpha \alpha)} \cap B_+$ (by assumption) $= (B_+)_{\gamma(n_\alpha \alpha)}$. (This makes use of the fact that $\gamma \circ \pi = \pi \circ \gamma$ on G .) Hence $\gamma(n_\alpha \alpha) > 0 \forall \alpha, \gamma$; now use the first part. \square

We now introduce the following notation: for $y \in Y$, we write $y = (\lambda_y, M_y, J_y)$ (see equation (5.1)). This representation of y may not be unique, e.g., under the action of Γ . We also define *Verma modules* to be $Z(y) = \Delta(M_y)$, for $y \in Y$. The next result is standard; the first part uses Standing Assumption 6.1, via (the second part of) Lemma 6.1.

Proposition 6.2.

- (1) If $y \in Y$ and $(M_y)_\lambda \neq 0$, then $(Z(y))_\lambda = (M_y)_\lambda$, where $\lambda = \lambda_y$.
- (2) If $x \in X$, then $Z(x)$ has a unique simple quotient $V(x)$. Its “highest weight” vectors also span M_x (as is true in $Z(x)$).
- (3) $Z(x)$ is indecomposable for $x \in X$.

We next turn to standard cyclic modules, namely, modules generated by a single *maximal* (i.e., in $\ker N_+$) weight vector.

Definition 6.1.

- (1) A *standard cyclic module* is a quotient of $\Delta(M_y)$ for $y \in Y$.
- (2) A module M has an *SC-filtration* or a *p-filtration* (denoted by $M \in \mathcal{F}(\Delta)$; see [BGG]) if it has a finite filtration whose successive quotients are standard cyclic or Verma modules, respectively.
- (3) A module M has a *simple Verma flag* if it has a *p-filtration* by Verma modules $\{Z(x) : x \in X\}$.
- (4) We define a relation on Y : we say $y \leq y'$ if and only if $\lambda_y < \gamma(\lambda'_y)$ in G for some $\gamma \in \Gamma$, or else $y = y'$.

Proposition 6.3.

- (1) \leq is a partial order on Y .
- (2) If E is any finite-dimensional H -semisimple $(H \otimes B_+) \rtimes \Gamma$ -module, then $\text{Ind } E$ has a simple Verma flag.

Proof.

- (1) If $y \leq y'$ and $y' \leq y$, then $y = y'$ by Lemma 6.1.
- (2) Look at a composition series for E in \mathcal{C} , say $0 \subset E_1 \subset \cdots \subset E_n = E$, with $E_{i+1}/E_i \cong M_{x_i}$ for some $x_i \in X$. Using formal characters, we can rearrange the E_i ’s (see [BGG]), such that $x_i \geq x_j \Rightarrow i \leq j$.

Character theory now shows that this “rearranged” filtration is a chain of $(H \otimes B_+) \rtimes \Gamma$ -modules. But then $0 \subset \text{Ind } E_1 \subset \cdots \subset \text{Ind } E_n = \text{Ind } E$ is a simple Verma flag for $\text{Ind } E$, by the exactness of Ind (from Proposition 6.1 above). \square

7. SIMPLE MODULES

We now classify all simple modules in \mathcal{O} , as well as those of them which are finite-dimensional. We assume that we have classified all finite-dimensional simple A -modules $V_A(\lambda) \in \mathcal{O}_A$. (Note, as in [Kh1], that if k is algebraically closed, then all finite-dimensional simple A -modules are in \mathcal{O}_A , and hence are of the form $V_A(\lambda)$ for some λ .) The following is trivial.

Lemma 7.1. *Given $M \in \mathcal{O}$, a weight vector $v \in M$ is maximal, if and only if so is γv for any $\gamma \in \Gamma$.*

Proposition 7.1 (“Weyl Character Formula 1”). *Fix $x \in X$, and consider $M_x \subset V(x)$. Then $B_-v_\mu = V_A(\mu)$ for all weight vectors $v_\mu \in M_x$, and left-multiplication by γ is a vector space isomorphism $: V_A(\mu) \rightarrow V_A(\gamma(\mu))$. Thus, $\text{ch}_{V_A(\gamma(\mu))} = \gamma(\text{ch}_{V_A(\mu)})$ for all γ, μ .*

Moreover, if $\{\gamma_i v_{\lambda_x} : 1 \leq i \leq n\}$ is a weight basis of M_x , then $V(x) = \bigoplus_i V_A(\gamma_i(\lambda_x))$ as A -modules. In particular, $\text{ch}_{V(x)} = \sum_{i=1}^n \text{ch}_{V_A(\gamma_i(\lambda_x))}$.

Proof. We first note that if $B_-v_\mu = V_A(\mu)$ for all $v_\mu \in M_x$, then $\gamma B_-v_\mu = \gamma B_- \gamma^{-1} \cdot \gamma v_\mu = B_- \gamma v_\mu$, and this must equal $V_A(\gamma(\mu))$ (it is simple because any maximal vector in $\gamma V_A(\mu)$ must come from one in $V_A(\mu)$). Thus the second part follows from the first (and holds for every μ , since each μ is of the form λ_x for some $x \in X$, from above results).

Observe that v_μ is maximal, being in M_x , hence B_-v_μ is a standard cyclic module in \mathcal{O}_A . We claim that it is simple. Suppose not. Then there exists a maximal vector $b_-v_\mu \in B_-v_\mu$, of weight $\nu < \mu$, say. We now claim that $V := \sum_\gamma B_- \gamma \cdot (b_-v_\mu) = \sum_\gamma \gamma B_-(b_-v_\mu)$ is a (nonzero) proper $(A \rtimes \Gamma)$ -submodule of $V(x)$ (which is a contradiction). For since the γb_-v_μ ’s have weights $\gamma(\nu) < \gamma(\mu)$ respectively, are maximal vectors by Lemma 7.1, and generate V , hence $v_\mu \notin V$ by Lemma 6.1. Thus $B_-v_\mu = V_A(\mu)$, as claimed.

Finally, if $\{\gamma_i v_{\lambda_x}\}$ is a basis of M_x , then

$$V(x) = B_-M_x = B_-(k\Gamma \cdot v_{\lambda_x}) = \sum_i B_- \gamma_i v_{\lambda_x} = \sum_i V_A(\gamma_i(\lambda_x))$$

Now, $\gamma_i(v_{\lambda_x}) \notin V' := \sum_{j \neq i} V_A(\gamma_j(\lambda_x))$, so $V_A(\gamma_i(\lambda_x)) \cap V' = 0$. Hence the above sum of simple A -modules is direct; now use character theory. \square

Remark 7.1. The same results hold if we replace simple modules by Verma modules, i.e., $x \in X$ by $y \in Y$, V ’s by Z ’s, and V_A ’s by Z_A ’s respectively (using the proof of Proposition 6.1). Moreover, we claim that the length $l_A(Z_A(\gamma(\lambda)))$ (if finite) is independent of γ , for any $\lambda \in G$. For, choose any $x \in X$ such that $\lambda_x = \lambda$; then $B_- \gamma v_\lambda \cong Z_A(\gamma(\lambda))$ as A -modules, for all γ .

Moreover, $\gamma : Z_A(\lambda) \rightarrow Z_A(\gamma(\lambda))$ takes maximal vectors to maximal vectors, so it preserves the length of any filtration.

Corollary 7.1. *Suppose $V_A(\lambda)$ is finite-dimensional. Then so is $V_A(\gamma(\lambda))$ for all γ , and the dimension is independent of γ . Moreover, $\dim V(x) = \dim M_x \cdot \dim V_A(\lambda)$, if $\lambda = \lambda_x$.*

Proof. For all $\mu \in G$, $\gamma : V_A(\mu) \rightarrow V_A(\gamma(\mu))$ is a vector space isomorphism (as A -submodules of some $V(x)$), by Theorem 5.1 and Proposition 7.1. The next part is also clear by Proposition 7.1. \square

All these facts come together in proving

Theorem 7.1.

- (1) *Every simple module in \mathcal{O} is of the form $V(x)$ for some $x \in X$.*
- (2) *Given $x \in X$, the simple module $V(x)$ is finite-dimensional if and only if $V_A(\lambda_x)$ is finite-dimensional.*
- (3) *Given $x \in X$, the Verma module $Z(x)$ is of finite length, if and only if $Z_A(\lambda_x) \in \mathcal{O}_A$ is.*

Proof.

- (1) This is standard from the previous section, say.
- (2) If $V(x)$ is finite-dimensional, then so is its A_i -submodule $V_A(\lambda_x)$ (by Proposition 7.1). The converse follows from Corollary 7.1.
- (3) If each $Z_A(\lambda_x) \in \mathcal{O}_A$ has finite length, then by Remark 7.1, $Z(x)$ has finite length as an A -module - hence also as an $A \rtimes \Gamma$ -module. Conversely, if $l_{A \rtimes \Gamma}(Z(x)) < \infty$, then $l_A(Z(x)) < \infty$, since by Proposition 7.1, $l_A(V(x')) < \infty \forall x'$. Now use Remark 7.1 again. \square

Corollary 7.2. *If k is algebraically closed, then every finite-dimensional simple $A \rtimes \Gamma$ -module V is of the form $V(x) \in \mathcal{O}$ for some $x \in X$.*

8. DUALITY

We now introduce the duality functor, that helps obtain information about the Ext-quiver in \mathcal{O} (and its relation to the partial order on the simple objects). We first make a general definition. Suppose we have a k -algebra A' , satisfying the following:

There exists an anti-involution $i : A' \rightarrow A'$, that fixes a subalgebra $H' \subset A'$.

Definition 8.1.

- (1) The *Harish-Chandra category* $\mathcal{H}' = \mathcal{H}_{A', H'}$ over (A', H') consists of all A' -modules M with a simultaneous weight space decomposition for H' , and finite-dimensional weight spaces.
- (2) The *duality functor* $F : \mathcal{H}' \rightarrow \mathcal{H}'$ is defined as follows: $F(M)$ is the span of all H' -weight vectors in $M^* = \text{Hom}_k(M, k)$. It is a module under: $\langle a'm^*, m \rangle = \langle m^*, i(a')m \rangle$ for $a' \in A', m \in M, m^* \in F(M)$.

Remark 8.1. One can then show [Kh1, Propositions 1,2] (except for part 2 of Proposition (2.2)):

Proposition 8.1. *F is exact, contravariant, and preserves lengths and formal characters. Moreover, $F(F(M)) = M$ for all $M \in \mathcal{H}'$.*

8.1. Functoriality. Now suppose we have algebras $A' \supset A'' \supset H'$, with $i_{A'}|_{A''} = i_{A''}$. We then have the duality functors F', F'' on the Harish-Chandra categories $\mathcal{H}' = \mathcal{H}_{A', H'}, \mathcal{H}'' = \mathcal{H}_{A'', H'}$ respectively, and the forgetful functor $\text{id}' : \mathcal{H}' \rightarrow \mathcal{H}''$. The following is easy to prove.

Lemma 8.1. *$F'' \circ \text{id}' = \text{id}' \circ F'$ on \mathcal{H}' .*

8.2. Application to the BGG Category. Note that $A \rtimes \Gamma$ has an anti-involution $i = i_A \otimes i_\Gamma$ by the standing assumptions. This enables us to define the duality functor $F : \mathcal{O} \rightarrow \mathcal{O}^{op} \subset \mathcal{H}$, as in [Kh1]. Now, F permutes the set of simple objects, so $F(V(x))$ is also a simple object in $\mathcal{H} = \mathcal{H}_{A \rtimes \Gamma, H}$. Moreover, Γ acts on formal characters (i.e., on $\mathbb{Z}[G]$): if $e(\lambda) \in \mathbb{Z}[G]$ corresponds to $\lambda \in G$, then $\gamma(e(\lambda)) = e(\gamma(\lambda))$.

We now put $A' = H \rtimes \Gamma$ and $H' = H$. Then the analogous results hold, and we get a duality functor on $\mathcal{H}_{H \rtimes \Gamma}$, that restricts to one on \mathcal{C} as well. In particular, F permutes the set of simple objects, i.e., $F : X \rightarrow X$. For example, if $\Gamma = 1$, then $F(\lambda) = \lambda \in G$, since each $x \in X = G$ is then one-dimensional.

The following result relates the dualities in \mathcal{C} and \mathcal{O} , and generalizes part 2 of [Kh1, Proposition 2.2].

Proposition 8.2. *For all $x \in X$, we have $F(V(x)) = V(F(x))$.*

Proof. We know that $F(V(x))$ is a simple module in \mathcal{H} with the same formal character as $V(x)$. It thus has a weight vector of maximal weight, which generates the entire module, since it is simple. Thus $F(V(x)) \in \mathcal{O}$, whence it is of the form $V(x')$ from above. We claim that $x' = F(x)$.

To see this, apply Lemma 8.1, setting $A' = A \rtimes \Gamma, A'' = H \rtimes \Gamma, H' = H$. The “highest” set of weight spaces in $V(x)$ is the $H \rtimes \Gamma$ -module M_x , and F'' sends it to $M_{F(x)}$; now by Lemma 8.1, $F(V(x)) = V(F(x))$. \square

We conclude with a standard variant of Schur’s Lemma, in addition to dualizing the fact that $V(x)$ is the unique simple quotient of $Z(x)$.

Proposition 8.3.

- (1) *For all $x \in X$, $F(Z(x))$ has socle $V(F(x))$.*
- (2) *Given $x, x' \in X$,*

$$\begin{aligned}
 & \text{Hom}_{A \rtimes \Gamma}(Z(x), F(Z(F(x')))) = \text{Hom}_{A \rtimes \Gamma}(Z(x), V(x')) \\
 &= \text{Hom}_{(H \otimes B_+) \rtimes \Gamma}(Z(x), M_{x'}) = \text{Hom}_{(H \otimes B_+) \rtimes \Gamma}(V(x), M_{x'}) \\
 &= \text{Hom}_{(H \otimes B_+) \rtimes \Gamma}(M_x, M_{x'}) = \text{Hom}_{(H \otimes B_+) \rtimes \Gamma}(M_x, V(x')) \\
 &= \text{Hom}_{A \rtimes \Gamma}(V(x), V(x')) = \delta_{x, x'} \text{End}_{H \rtimes \Gamma}(M_x),
 \end{aligned}$$

where each M_x is killed by N_+ , and $M \twoheadrightarrow M_x$ for $M = Z(x), V(x)$, with $\text{kernel}(s) \oplus_{\mu < \gamma(\lambda_x)} M_\mu$.

- (3) $\dim_k \text{End}(M_x) = \dim_k \text{End}(M_{F(x)}) \ \forall x \in X$.
- (4) If k is algebraically closed when $|\Gamma| > 1$, then $M_x, V(x)$, and $Z(x)$ are Schurian, i.e., the only module endomorphisms are scalars.

We merely remark that in the second part, the first equality is standard. Moreover, for all terms but the first and the last, all Hom-groups are clearly zero unless $x = x'$, in which case, we produce a cyclic chain of maps $\varphi_{AB} : A(x) \rightarrow B(x')$, that compose to give the identity:

$$\varphi_{ZV} \mapsto \varphi_{ZM} \mapsto \varphi_{VM} \mapsto \varphi_{MM} \mapsto \varphi_{MV} \mapsto \varphi_{VV} \mapsto \varphi_{ZV}.$$

9. HOMOLOGICAL PROPERTIES

In this section, we show some results that are needed in later sections.

9.1. Every object has a good filtration. We now show that every module in \mathcal{O} has a filtration with standard cyclic subquotients. As in [Kh1], we need more notation.

Definition 9.1.

- (1) Define $\text{ht} : \mathbb{Z}\Delta \rightarrow \mathbb{Z}$, via: $\text{ht}(\sum_{\alpha \in \Delta} n_\alpha \alpha) := \sum_{\alpha} n_\alpha$.
- (2) Given $l \in \mathbb{Z}_{\geq 0}$, define $B_{+l} := \sum_{\text{ht } \pi(\theta) \geq l} (B_+)_{\theta}$ in B_+ for all $l \in \mathbb{N}$.
- (3) Given $\lambda \in G$ and $l \in \mathbb{N}$, define the subcategory $\mathcal{O}(\lambda, l)$ to be the full subcategory of all $M \in \mathcal{O}$ such that $B_{+l} \cdot k\Gamma \cdot M_\lambda = 0$.
- (4) Given $y \in Y$, define the module

$$P(y, l) := (A \rtimes \Gamma) / (B_{+l}, J_y),$$

where $y = (\lambda_y, M_y, [J_y])$, and we identify 1 with the generator v_{λ_y} (by choice of $[J_y]$), as in equation (5.1). (Thus $h - \lambda_y(h) \cdot 1 \in J_y$ for all $h \in H$.)

- (5) Define $I_{y,l}$ to be the left ideal of (the k -algebra) $(H \otimes B_+) \rtimes \Gamma$ generated by B_{+l} and J_y , and set $E_{y,l} := ((H \otimes B_+) \rtimes \Gamma) / I_{y,l}$.

Thus, $\text{Ind } E_{y,l} = B_- \otimes E_{y,l} = (A \rtimes \Gamma) / ((A \rtimes \Gamma) I_{y,l}) = P(y, l)$; in particular, $P(y, l) \in \mathcal{O}$. In fact, by considering the formal character of $E_{y,l}$, $P(y, l) \in \mathcal{O}(\lambda_y, m) \ \forall m \geq l, y \in Y$; moreover, $P(y, l+1) \twoheadrightarrow P(y, l) \ \forall y, l$.

Note that $k\Gamma \cdot \bar{1} = M_y \subset E_{y,l}$. Since $E_{y,l} = B_+ M_y$ is a finite-dimensional H -semisimple $(H \otimes B_+) \rtimes \Gamma$ -module, hence this is similar to a construction in [BGG], and by the above results, $P(y, l)$ has a simple Verma flag, with a “suitable arrangement” of composition factors, by the proof of Proposition 6.3 above. In fact, for all l , one of the terms in the Verma flag for $P(x, l)$ is $Z(x)$, and by inspecting the formal character of $E_{x,l}$ (or its direct sum decomposition as a $H \rtimes \Gamma$ -module), we find that all other terms are of the form $Z(x')$ for $x' > x$. For example, if $l = 1$, then $B_{+1} = N_+$, so $P(y, 1) = Z(y)$.

We now analyze singly-generated modules in \mathcal{O} , and then all modules. If $N = (A \rtimes \Gamma)v_\lambda$ for some (not necessarily maximal) weight vector v_λ , then $N = B_-B_+(k\Gamma v_\lambda)$, where we note that $k\Gamma v_\lambda = M_y$, say, for some $y \in Y$. Since $N \in \mathcal{O}$, hence $B_+(k\Gamma v_\lambda)$ is finite-dimensional, so that $P(y, l) \twoheadrightarrow N = (A \rtimes \Gamma)v_{\lambda_y}$ for all $l \gg 0$. In fact, we have:

Proposition 9.1.

- (1) For all $l \geq 0$, B_{+l} is a two-sided ideal of B_+ with finite codimension, that is stable under the Γ -action.
- (2) If $N \in \mathcal{O}(\lambda, l)$ and $\lambda = \lambda_y$, then $\text{Hom}_{\mathcal{O}}(P(y, l), N) = \text{Hom}_{H \rtimes \Gamma}(M_y, k\Gamma \cdot N_{\lambda_y})$.
- (3) $P(y, l)$ is projective in $\mathcal{O}(\lambda_y, l)$ for all $y \in Y$, $l \in \mathbb{N}$.
- (4) If $N \in \mathcal{H}$, then the following are equivalent:
 - (a) $N \in \mathcal{O}$.
 - (b) N is a quotient of a (finite) direct sum of $P(y_i, l_i)$'s.
 - (c) N has an SC-filtration, with subquotients of the form $Z(x) \rightarrow V \rightarrow 0$, with $x \in X$.

Proof.

- (1) Two-sidedness follows from (RTA7), and finite codimension from (RTA6). Next, if $b \in B_{+l}$ is a weight vector of weight λ , then $\text{ht } \lambda \geq l$. But then by Lemma 6.1, $\text{ht } \gamma(\lambda) \geq l$, whence $\gamma(b) \in B_{+l}$.
- (2) This is as in [BGG] (or also [Kh1, Proposition 5]).
- (3) This follows from the previous part and Proposition 5.1 above.
- (4) This is proved as in [Kh1, Proposition 7].

□

A standard consequence is

Proposition 9.2. *The following are equivalent:*

- (1) $Z_A(\lambda)$ has finite length for all λ .
- (2) $Z(x)$ has finite length (as an $A \rtimes \Gamma$ -module) for all $x \in X$.
- (3) $Z(y)$ has finite length for all $y \in Y$.
- (4) \mathcal{O}_A is finite length.
- (5) $\mathcal{O}_{A \rtimes \Gamma}$ is finite length.

9.2. Extensions between simple objects. We now obtain information about the Ext-quiver in \mathcal{O} . Recall the partial ordering on Y , namely: $y \leq y'$ if and only if $\lambda_y < \gamma(\lambda_{y'})$ for some $\gamma \in \Gamma$, or $y = y'$. We also define $Y(x)$ (for any x) to be the unique maximal submodule of $Z(x)$. We now imitate a result in [Kh2]; the proof is similar to that of [Kh1, Proposition 4, part 2].

Proposition 9.3. $E_{x,x'} := \text{Ext}_{\mathcal{O}}^1(V(x), V(x'))$ is nonzero if and only if $(x > x' \text{ and } Y(x) \twoheadrightarrow V(x'))$, or $(x' > x \text{ and } Y(F(x')) \twoheadrightarrow V(F(x)))$. Moreover, $E_{x,x'} \cong E_{F(x'), F(x)}$ is finite-dimensional.

We now show our first block decomposition - though not for \mathcal{O} . The proof uses Proposition 9.3 and an argument similar to [Kh1, Theorem 4]; each “finite length block” is clearly self-dual.

Proposition 9.4. *Define $\mathcal{O}_{\mathbb{N}}$ to be the full subcategory of all finite length objects in \mathcal{O} . The sum $\sum_{x \in X} (\mathcal{O}(x) \cap \mathcal{O}_{\mathbb{N}})$ of distinct subcategories (of all finite length objects in $\mathcal{O}(x)$) is direct, and equals all of $\mathcal{O}_{\mathbb{N}}$. Each “finite length block” is abelian, (finite length,) self-dual, and closed under morphisms and extensions (in $\mathcal{O}_{\mathbb{N}}$). All morphisms and extensions between distinct blocks are trivial.*

10. THE CONDITIONS (S)

Proposition 10.1. *Fix $y \in Y, \mu \in G$. The following are equivalent:*

- (1) $[Z_A(\gamma(\lambda_y)) : V(\mu)] > 0$ for some $\gamma \in \Gamma$.
- (2) *There exists $x \in X$ so that $\lambda_x = \mu$ and $[Z(y) : V(x)] > 0$.*

For this result (and also later), we need the following lemma.

Lemma 10.1. *For any ring R , every simple (sub)quotient of a direct sum of R -modules, is automatically a simple (sub)quotient of some summand.*

Proof of the proposition. First assume that (1) holds; then there is a $b_- \in B_-$ with $b_- w_{\gamma(\lambda_y)} = w_\mu$, where $w_{\gamma(\lambda_y)}, w_\mu$ are maximal weight vectors of appropriate weights. Now assume v_{λ_y} is maximal in $Z(y)$. Then we claim that $v_\mu := b_- \gamma v_{\lambda_y}$ is maximal in $Z(y)$; this is because $B_- \cdot \gamma v_{\lambda_y} \cong Z_A(\gamma(\lambda_y))$ as A -modules.

Let us now write $k\Gamma \cdot v_\mu$ as a direct sum of simple $(H \otimes B_+) \rtimes \Gamma$ -modules, by results above. If M_x is any simple summand, then we see that $V(x)$ is a subquotient of $Z(y)$, with $\lambda_x = \mu$; thus, (2) follows.

Conversely, assume (2). Given a weight basis \mathcal{B} of $M_y \subset Z(y)$, we have $Z(y) \cong \bigoplus_{v \in \mathcal{B}} B_- v$ as A -modules (from Remark 7.1), with each summand a Verma module for A . So if $[Z(y) : V(x)] > 0$, then as A -modules, $V_A(\lambda_x)$ is a simple subquotient of $Z(y)$. By Lemma 10.1, $V_A(\lambda_x)$ is a simple subquotient of some $B_- v$, hence of $Z_A(\gamma(\lambda_y))$ for some γ . \square

Next, recall the definition of $S^n(\lambda) = S_A^n(\lambda), S^n(x)$, for $\lambda \in G$ and $x \in x$ (see Definition 4.1). We now relate the Conditions (S) for A and $A \rtimes \Gamma$.

Proposition 10.2. *Define $S_\Gamma(x) := \{x' \in X : \lambda_{x'} = \gamma(\mu) \text{ for some } \gamma \in \Gamma, \mu \in S_A^3(\lambda_x)\}$. Now given all $\lambda \in G, \gamma \in \Gamma, x \in X$, we have:*

- (1) $\gamma(S_A^n(\lambda)) = S_A^n(\gamma(\lambda))$ for $n = 1, 2, 3$.
- (2) *The map $\text{wt} : X \rightarrow G/\Gamma$, sending $x \mapsto \lambda_x$, satisfies:*

$$\bigcup_{\text{wt}(x)=\lambda} S^3(x) = S_\Gamma(x) = \text{wt}^{-1}(\Gamma(S_A^3(\lambda))) \quad \forall x \in \text{wt}^{-1}(\lambda). \quad (10.1)$$

- (3) *For $n = 1, 2$, and any $\lambda \in G$,*

$$\bigcup_{\text{wt}(x)=\lambda} S^n(x) = \Gamma(S_A^n(\lambda)). \quad (10.2)$$

Note that in both equations, only the left-hand side (supposedly) depends on the specific λ inside an “ S^n -set”.

Proof.

- (1) We first show that γ takes “edges” in $S_A^3(\lambda)$ to “edges” in $S_A^3(\gamma(\lambda))$. Take any $y \in Y$ such that $\lambda = \lambda_y$ (such a y exists, by Remark 5.1). Now consider the Verma module $Z(y)$. For all $v_{\gamma(\lambda_y)} = \gamma v_{\lambda_y} \in M_y$, we have the A -Verma module $Z_A(\gamma(\lambda_y)) = B_- v_{\gamma(\lambda_y)}$. Now use Remark 7.1 to observe that $V_A(\mu)$ is a subquotient of $Z_A(\lambda) \subset Z(y)$ if and only if $V_A(\gamma(\mu))$ is a subquotient of $Z_A(\gamma(\lambda))$. (We note that $v_\mu \in Z_A(\lambda)$ is maximal if and only if so is γv_μ , by Lemma 7.1.)

The proof is similar if $[Z_A(\mu) : V_A(\lambda)] > 0$: start with $y \in Y$ so that $\mu = \lambda_y$. Applying transitivity, we conclude that $\gamma(S_A^3(\lambda)) \subset S_A^3(\gamma(\lambda))$. Replacing γ by γ^{-1} and $\lambda = \lambda_y$ by $\gamma(\lambda)$, we get:

$$S_A^3(\lambda) = \gamma^{-1}(\gamma(S_A^3(\lambda))) \subset \gamma^{-1}(S_A^3(\gamma(\lambda))) \subset S_A^3(\gamma^{-1}(\gamma(\lambda))) = S_A^3(\lambda).$$

This proves the result for (S3). The other two subparts now follow, by using this subpart and Lemma 2.1:

$$\begin{aligned} \gamma(S_A^2(\lambda)) &= \gamma(\pi(S_A^3(\lambda))) = \pi(\gamma(S_A^3(\lambda))) = \pi(S_A^3(\gamma(\lambda))) = S_A^2(\gamma(\lambda)), \\ \gamma(S_A^1(\lambda)) &= \gamma(S_A^2(\lambda)^{\leq \lambda}) = S_A^2(\gamma(\lambda))^{\leq \gamma(\lambda)} = S_A^1(\gamma(\lambda)). \end{aligned}$$

- (2) We show this in two parts.

- (a) It is clear that $\text{wt}^{-1}(S_A^3(\lambda)) = S_\Gamma(x)$ if $\lambda_x = \lambda$. Next, if $\lambda_x \in \Gamma(\lambda)$, then we claim that $S^3(x) \subset \text{wt}^{-1}(\Gamma(S_A^3(\lambda)))$:

First, $\lambda_{F(x)} = \lambda_x \in S_A^3(\lambda_x) \forall x$. Next, if $[Z(x) : V(x')] > 0$, then using the structure of $V(x')$, $Z(x)$, we see that $V_A(\lambda_{x'})$ is a subquotient of a direct sum of some $Z_A(\gamma(\lambda_x))$'s. By the previous part, Lemma 10.1, and Proposition 10.1, $\lambda_{x'} \in S_A^3(\gamma(\lambda_x)) = \gamma(S_A^3(\lambda_x))$. Hence $\lambda_{x'} \in \Gamma(S_A^3(\lambda_x))$, and by symmetry and transitivity, the claim is proved.

- (b) We now show that $\text{wt}^{-1}(\Gamma(S_A^3(\lambda))) \subset \bigcup_{\text{wt}(x)=\lambda} S^3(x)$. Recall the graph structure on G , under which $S_A^3(\lambda)$ is a connected component. We now prove this inclusion by induction on $d(-, \lambda)$, where $d(-, -)$ is the *graph distance function* (and λ is the distinguished weight in the statement).

In what follows, λ_k will denote some element of $S_A^3(\lambda)$ such that $d(\lambda_k, \lambda) = k \geq 0$. (This is well-defined, since $\gamma : S_A^3(\lambda) \rightarrow S_A^3(\gamma(\lambda))$ “takes edges to edges” by the previous part.)

The result is clear for $k = 0$, since $\lambda_0 = \lambda$. Now suppose that it holds for all λ_k (for some fixed $k \geq 0$). Given $\lambda_{k+1} \in S_A^3(\lambda)$, we know there exists a λ_k connected to it by an edge. Suppose $[Z(\lambda_{k+1}) : V(\lambda_k)] > 0$ (the proof of the other case is similar). Choose any $x_{k+1} \in \text{wt}^{-1}(\lambda_{k+1})$; then by Proposition 10.1, there exists $x_k \in \text{wt}^{-1}(\lambda_k) \subset \bigcup_{\text{wt}(x)=\lambda} S^3(x)$ (by the induction hypothesis), such that $[Z(x_{k+1}) : V(x_k)] > 0$. But then $x_{k+1} \in \bigcup_{\text{wt}(x)=\lambda} S^3(x)$ as well, and we are done by induction.

- (3) By equation (10.1), $\bigcup_{\text{wt}(x)=\lambda} \text{wt}(S^3(x)) = \Gamma(S_A^3(\lambda))$. To prove equation (10.2) for $n = 2$, apply π to both sides and use Lemma 2.1. To prove the equation for $n = 1$, first intersect both sides with $G^{\leq \lambda}$, and then apply π . We are then done, if we note that

$$\bigcup_{\text{wt}(x)=\lambda} S^3(x)^{\leq \lambda} = \bigcup_{\text{wt}(x)=\lambda} S^3(x)^{\leq x}.$$

□

We now collect our results relating the Conditions (S) for A and $A \rtimes \Gamma$.

Theorem 10.1. *Suppose $A \rtimes \Gamma$ is a skew group ring over an RTA.*

- (1) *The various Conditions (S) satisfy: (S3) \Rightarrow (S2) \Rightarrow (S1).*
- (2) *Suppose k is algebraically closed whenever $|\Gamma| > 1$. Then $A \rtimes \Gamma$ satisfies each of the Conditions (S1), (S2), (S3), if and only if A does.*
- (3) *If Condition (S1) holds, then \mathcal{O} is finite length.*
- (4) *If \mathcal{O} is finite length, then it splits into a direct sum of blocks.*

Note that semisimple Lie algebras and quantum groups satisfy Condition (S4) (as seen earlier), hence all the Conditions (S) as well - and hence if k is algebraically closed of characteristic zero, then their wreath products also satisfy all the Conditions (S). (We relate Condition (S4) across the various setups, in Section 11.)

Proof. The first part is by definition, and the last part follows from Proposition 9.4. The second part follows from equations (10.1) and (10.2), and Theorem 5.1, because (if k is algebraically closed,) wt is a finite-to-one map.

We now abuse notation (using the freeness of the action $*$) to write $\lambda - \lambda' = \theta$ when $\theta * \lambda' = \lambda$, and use that π intertwines the $*$ -actions on G and G_0 . Now given $x \in X$, a submodule (or a maximal vector) in $Z(x)$ of restricted highest weight $\lambda_0 (\neq \pi(\gamma(\lambda_x))) \in G_0$ occurs only if $\lambda_0 = \pi(\lambda_{x'})$ for some $x' < x$ in $S^3(x)$. This yields:

$$l_{A \rtimes \Gamma}(Z(x)) \leq \sum_{\lambda \in S^1(x)} p(\lambda_0 - \pi(\lambda_x)),$$

where p is the *Kostant partition function* : $G \rightarrow \mathbb{Z}_{\geq 0}$, defined by $p(\theta) := \dim_k(B_-)_\theta$. But this summation is a finite sum by assumption, and each summand is finite by regularity of A ; now use Proposition 9.2. □

11. CENTRAL CHARACTERS

Next, we discuss the notion of *central characters*, as in [BGG] - but over skew group rings now. All but the last subsection are general; the last focusses on $R_{\mathfrak{g}} = S_n \wr \mathfrak{U}\mathfrak{g}$.

11.1. Central characters for skew group rings. We start with a few definitions. Given a skew group ring $A \rtimes \Gamma$ over an RTA A (satisfying Standing Assumptions 2.1, 6.1), recall the triangular decomposition $A \cong B_- \otimes H \otimes B_+$, the augmentation ideals N_\pm of B_\pm , and the (possibly infinite) sets $S_A^3(\lambda)$ for each $\lambda \in G$, defined above.

Definition 11.1. To start with, denote the center of $A \rtimes \Gamma$ by Z .

- (1) The *Harish-Chandra projection* is $\xi := \epsilon^- \otimes \text{id} \otimes \epsilon^+ : A = B_- \otimes H \otimes B_+ \rightarrow H$ (see Proposition 2.1).
- (2) Given $\lambda \in G$, the subgroups $\Gamma^\lambda, \Gamma^{S_A^3(\lambda)}$ of Γ are defined to be: $\Gamma^\lambda := \{\gamma \in \Gamma : \gamma(\lambda) = \lambda\}$, $\Gamma^{S_A^3(\lambda)} := \bigcap_{\mu \in S_A^3(\lambda)} \Gamma^\mu$.
- (3) Given $\lambda \in G$, the *central character* χ_λ is the map $: A \rtimes \Gamma \rightarrow k\Gamma$, defined as follows: given $r = \sum_{\gamma \in \Gamma} a_\gamma \gamma$ with $a_\gamma \in A$, define $\chi_\lambda(r) := \sum_{\gamma \in \Gamma^\lambda} \lambda(\xi(a_\gamma)) \gamma$.

This definition is motivated by the fact that in [BGG] (where $|\Gamma| = 1$), the center acts via such characters on (maximal vectors in) objects in \mathcal{O} .

Lemma 11.1. *If v_λ is a maximal vector (i.e., $N_+ v_\lambda = 0$) of weight λ in any $A \rtimes \Gamma$ -module M , then $z v_\lambda = \chi_\lambda(z) v_\lambda$ in M , for all $z = \sum_\gamma z_\gamma \gamma$ in Z .*

Proof. Since each γv_λ is maximal, hence writing $z_\gamma = n_\gamma \oplus b_\gamma \oplus \xi(z_\gamma)$ (with $n_\gamma \in AN_+, b_\gamma \in N_- H$), we get that n_γ kills γv_λ , and $b_\gamma \gamma v_\lambda$ has no λ -weight component, by Lemma 6.1. Hence:

$$z v_\lambda = \sum_{\gamma \in \Gamma} \xi(z_\gamma) \gamma v_\lambda = \sum_{\gamma \in \Gamma} \gamma(\lambda) (\xi(z_\gamma)) \gamma v_\lambda.$$

This is because if $v_\lambda \in M_\lambda$ for any $A \rtimes \Gamma$ -module M , then we claim that $z v_\lambda \in M_\lambda$ as well: for any $h \in H$, we have $h \cdot z v_\lambda = z \cdot h v_\lambda = \lambda(h) z v_\lambda$. Thus, the above summation only runs over $\gamma \in \Gamma^\lambda$, so we have

$$z v_\lambda = \sum_{\gamma \in \Gamma^\lambda} \lambda(\xi(z_\gamma)) \gamma v_\lambda = \chi_\lambda(z) v_\lambda. \quad (11.1)$$

□

We now explore properties of central characters. The first result is that they are compatible with the Γ -action, in the following sense:

Proposition 11.1. *Given $r \in A \rtimes \Gamma, \lambda \in G, \beta \in \Gamma$, we have*

$$\chi_{\beta(\lambda)}(r) = \beta \chi_\lambda(\beta^{-1}(r)) \beta^{-1}.$$

Proof. Suppose $r = \sum_{\gamma \in \Gamma} a_\gamma \gamma$, with $a_\gamma \in A \forall \gamma$. Then

$$\chi_{\beta(\lambda)}(r) = \sum_{\gamma \in \Gamma^{\beta(\lambda)}} \langle \beta(\lambda), \xi(a_\gamma) \rangle \gamma = \sum_{\gamma \in \Gamma^{\beta(\lambda)}} \langle \lambda, \xi(\beta^{-1}(a_\gamma)) \rangle \gamma,$$

where the second equality holds because $\gamma(\xi(a)) = \xi(\gamma(a))$ for all $a \in A, \gamma \in \Gamma$ (since $\text{Ad } \gamma$ preserves N_{\pm}). Therefore

$$\begin{aligned} \chi_{\lambda}(\beta^{-1}(r)) &= \chi_{\lambda} \left(\sum_{\gamma \in \Gamma} \beta^{-1}(a_{\gamma}) \cdot \beta^{-1} \gamma \beta \right) = \sum_{\beta^{-1} \gamma \beta \in \Gamma^{\lambda}} \lambda(\xi(\beta^{-1}(a_{\gamma}))) \beta^{-1} \gamma \beta \\ &= \beta^{-1} \cdot \sum_{\gamma \in \Gamma^{\beta(\lambda)}} \lambda(\xi(\beta^{-1}(a_{\gamma}))) \gamma \cdot \beta = \beta^{-1} \chi_{\beta(\lambda)}(r) \beta \end{aligned}$$

using the above equation. \square

Next, we show that at least on the center, central characters have very few nonzero components:

Proposition 11.2. *Fix $\lambda \in G$. If $\gamma \notin \Gamma^{S_A^3(\lambda)}$, and $z = \sum_{\gamma} z_{\gamma} \gamma$ is any central element, then $\lambda(\xi(z_{\gamma})) = 0$.*

Proof. Consider equation (11.1), with $k\Gamma \cdot v_{\lambda} = M_y$, where $y = \text{Ind}_H^{H \rtimes \Gamma} k^{\lambda}$; thus $\dim_k y = |\Gamma|$. If $[Z_A(\lambda) : V_A(\mu)] > 0$ for some $\mu \in G$, then there is a maximal vector $v_{\mu} = bv_{\lambda}$ in $Z(y) = (A \rtimes \Gamma)v_{\lambda}$, for some weight vector $b \in N_-$ (e.g., by Proposition 10.1). We now compute using equation (11.1), for each $z \in Z$:

$$\sum_{\gamma \in \Gamma^{\lambda}} \lambda(\xi(z_{\gamma})) b \cdot \gamma v_{\lambda} = bz v_{\lambda} = zv_{\mu} = \sum_{\gamma \in \Gamma^{\mu}} \mu(\xi(z_{\gamma})) \gamma(b) \cdot \gamma v_{\lambda}. \quad (11.2)$$

By the triangular decomposition, the coefficient of γv_{λ} vanishes if $\gamma \notin \Gamma^{\lambda} \cap \Gamma^{\mu}$ (or $\gamma(b) \notin k^{\times} b$). Now apply symmetry and transitivity to get the result. \square

The above proof does not use the information about $\gamma(b) \notin k^{\times} b$; however, we will use it presently.

The main result in this subsection relates central characters to simple subquotients of Verma modules.

Theorem 11.1. *$Z = \mathfrak{Z}(A \rtimes \Gamma)$ as above.*

- (1) *For all $\lambda \in G$, χ_{λ} is an algebra map $: Z \rightarrow k\Gamma$.*
- (2) *To each $\lambda \in G$ is associated a subgroup Γ_{λ} of $\Gamma^{S_A^3(\lambda)}$, such that*
 - $\chi_{\lambda}(z) = \sum_{\gamma \in \Gamma_{\lambda}} \lambda(\xi(z_{\gamma})) \gamma \ \forall z = \sum_{\gamma} z_{\gamma} \gamma \in Z$;
 - $\Gamma_{\lambda} = \Gamma_{\mu}$ if $\mu \in S_A^3(\lambda)$; and
 - given $\mu \in S_A^3(\lambda)$, χ_{λ} and χ_{μ} are related (on Z) via a character $\theta_{\lambda, \mu}$ of Γ_{λ} : $\lambda(\xi(z_{\gamma})) = \theta_{\lambda, \mu}(\gamma) \mu(\xi(z_{\gamma})) \ \forall \gamma \in \Gamma_{\lambda}, z \in Z$.

Proof. To show the first part, choose any $z, z' \in Z$ and $y = \text{Ind}_H^{H \rtimes \Gamma} k^{\lambda}$. Then in the Verma module $Z(y) = (A \rtimes \Gamma)v_{\lambda}$, by Lemma 11.1,

$$\chi_{\lambda}(zz')v_{\lambda} = \chi_{\lambda}(z'z)v_{\lambda} = z'zv_{\lambda} = z'\chi_{\lambda}(z)v_{\lambda} = \chi_{\lambda}(z) \cdot z'v_{\lambda} = \chi_{\lambda}(z)\chi_{\lambda}(z')v_{\lambda},$$

because z' is central. The statement now follows, since y is the regular representation of Γ (as a Γ -module).

For the second part, we continue beyond the proof of Proposition 11.2. Thus, $\chi_\lambda(z) = \sum_{\gamma \in \Gamma^{S_A^3(\lambda)}} \lambda(\xi(z_\gamma))\gamma$, and the “improved” equation (11.2) suggests that $\lambda(\xi(z_\gamma)) = 0$ if $\gamma(b) \notin k^\times b$. Moreover, $\lambda(\xi(z_\gamma)) = 0$ if and only if $\mu(\xi(z_\gamma)) = 0$. Now denote $\Gamma_{\lambda,\mu} := \{\gamma \in \Gamma^{S_A^3(\lambda)} : \gamma(b) \in k^\times b \text{ for all } b \text{ such that } b\nu_\lambda \in Z(y)_\mu \text{ is maximal}\}$, and $\Gamma_\lambda := \bigcap \Gamma_{\nu',\nu''}$ for each $\nu' > \nu''$ in $S_A^3(\lambda)$ with $[Z_A(\nu') : V_A(\nu'')] > 0$ (though a more suitable name would be $\Gamma_{S_A^3(\lambda)}$). The first two sub-parts are now obvious, and by transitivity in $S_A^3(\lambda)$, it suffices to show the last sub-part when $[Z_A(\lambda) : V_A(\mu)] > 0$. But since Γ_λ acts on $k^\times \cdot b$, this yields a character $\theta_{\lambda,\mu}$ of Γ_λ ; now look at equation (11.2) again, and we are done. \square

Corollary 11.1. *If $\lambda \in G^\Gamma$, then χ_λ is an algebra map $: Z \rightarrow \mathfrak{Z}(k\Gamma)$.*

Proof. Use Proposition 11.1 and the first part of Theorem 11.1. \square

11.2. Another block decomposition. We now show a block decomposition of \mathcal{O} using central characters, under some extra assumptions.

Theorem 11.2.

- (1) *Suppose k is algebraically closed if $|\Gamma| > 1$. Then Z acts on Verma modules $Z(x)$ (for $x \in X$) via a central character, i.e., an algebra map $: Z \rightarrow k$, say χ_x . Then $\chi_x = \chi_{\lambda_x} = \chi_{\gamma(\lambda_x)}$ on $Z \cap A \forall \gamma \in \Gamma$.*
- (2) *Now suppose that Z acts on every $Z(x)$ via an algebra map $\theta_x : Z \rightarrow k$. Then $\mathcal{O} = \bigoplus_{\nu \in \Theta} \mathcal{O}(\nu)$, where ν runs over all distinct elements from among $\{\theta_x : x \in X\} \subset \text{Hom}_{k\text{-alg}}(Z, k)$ (call this set Θ), and $\mathcal{O}(\nu)$ is the full subcategory $\{M \in \mathcal{O} : \forall m \in M, z \in Z, \exists n \in \mathbb{N} \text{ such that } (z - \nu(z))^n \cdot m = 0\}$.*

Proof. The first part follows from Propositions 8.3 and 11.1, since $M_x(\subset Z(x))$ is Schurian and any $z \in Z$ is an endomorphism of M_x . For the second part, we use the following result:

Lemma 11.2. *Given $\nu \neq \nu'$ in Θ , $M \in \mathcal{O}(\nu')$, and $z \notin \ker(\nu - \nu')$, the map $\varphi := z - \nu(z)$ is an $A \rtimes \Gamma$ -module isomorphism on M .*

To see why, note that $\varphi = c + s$, where $s = z - \nu'(z)$ is locally nilpotent on M , and $c = \nu'(z) - \nu(z) \in k^\times$. Hence φ is invertible.

Given $M \in \mathcal{O}$, Proposition 9.1 implies (together with the first part) that there exist $\nu_i \in \Theta$ and $n_i \in \mathbb{N}$ such that $\prod_{i=1}^s (z - \nu_i(z))^{n_i}$ kills all of M , for all $z \in Z$. Now define $M(\nu) := \{m \in M : \forall z \in Z, \exists n \in \mathbb{N} \text{ such that } (z - \nu(z))^n m = 0\}$. Then this is an $A \rtimes \Gamma$ -submodule. Using Lemma 11.2, it is easy to see that each $\mathcal{O}(\nu)$ is closed in $A \rtimes \Gamma\text{-Mod}$ under taking submodules, quotients, and extensions, and all maps between distinct blocks are trivial. So if we show that $M = \bigoplus_\nu M(\nu)$, then each $M(\nu)$ is a quotient of $M \in \mathcal{O}$, hence also in \mathcal{O} , hence in $\mathcal{O}(\nu)$, and we will be done. To do this, we need the following standard lemma from commutative algebra.

Lemma 11.3. *If $\{\mathfrak{m}_i : 1 \leq i \leq s\}$ are distinct maximal ideals in Z for some $s \geq 1$, then $\sum_i \left(\prod_{j \neq i} \mathfrak{m}_j\right)^n = Z$ for all $n \geq 1$.*

Use the lemma to write $1 \in Z$ as $1 = \sum_{i=1}^s r_i$, with $r_i \in \left(\prod_{j \neq i} \ker \nu_j\right)^n$ and $n = \max_i n_i$ (s, ν_i, n_i as above). Then any $m \in M$ equals $\sum_i r_i m$, with $r_i m \in M(\nu_i)$. This shows that M is the sum of its components.

Finally, this sum is direct, for suppose $\sum_{i=1}^l m_i = 0$ for some $l > 1$, with $m_i \in M(\nu_i)$ for each i . If $i < l$, pick $z_i \notin \ker(\nu_i - \nu_l)$, and $n_i \in \mathbb{N}$ such that $(z_i - \nu_i(z_i))^{n_i}$ kills m_i . Then $\prod_{i < l} (z_i - \nu_i(z_i))^{n_i}$ kills m_1, \dots, m_{l-1} , whereas by Lemma 11.2, it is an isomorphism on $(A \rtimes \Gamma)m_l \subset M(\nu_l)$. Hence $m_l = 0$, and by induction on l , the other m_i 's vanish as well. \square

11.3. The Conditions (S), linking, and central characters. We now describe and relate different types of block decompositions. We need some definitions. We note that it is possible to obtain a block decomposition of \mathcal{O} using any of these sets.

Definition 11.2.

- (1) Define $CC(x) = S^4(x)$ to be the set of simple objects $\{x' \in X : \chi_{x'} = \chi_x\}$.
- (2) *Condition (S₄)* holds for $A \rtimes \Gamma$, if all sets $S^4(x)$ are finite.
- (3) We say that two indecomposable A -modules M, N are *linked* if $\text{Hom}_{A \rtimes \Gamma}(M, N) \neq 0$. Let the equivalence closure of M under such a relation be denoted by $[M]$.
- (4) Define $T(x)$ to be the equivalence closure of x in X , under the relations: $x \rightarrow F(x)$ and $x \rightarrow x'$ if $V(x') \in [V(x)]$.

(Note that we need the “intermediate modules” between linked modules to be indecomposable, otherwise any two modules are linked via: $M \rightarrow M \oplus N \rightarrow N$.) Also recall $S_\Gamma(x)$, wt from Proposition 10.2. We now compare these sets, and also mention a sufficient condition when the center is “nice” (this does indeed hold for wreath products).

Proposition 11.3.

- (1) *Given a skew group ring $A \rtimes \Gamma$, and $\lambda \in G, \gamma \in \Gamma$,*

$$\gamma(CC_A(\lambda)) = \gamma(S_A^4(\lambda)) = S_A^4(\gamma(\lambda)) = CC_A(\gamma(\lambda)).$$

- (2) $\mathfrak{Z}(A \rtimes \Gamma) \cap A = \mathfrak{Z}(A)^\Gamma$, *and given $\lambda \in G$,*

$$\{\mu \in G : \chi_\lambda \equiv \chi_\mu \text{ on } \mathfrak{Z}(A)^\Gamma\} = \Gamma(S_A^4(\lambda)).$$

- (3) *Suppose A is an integral domain, and each $\gamma \neq 1 \in \Gamma$ nontrivially permutes the restricted (i.e., G_0 -)weight spaces of A . Then $\mathfrak{Z}(A \rtimes \Gamma) = \mathfrak{Z}(A)^\Gamma = e_\Gamma \mathfrak{Z}(A) e_\Gamma$.*

Now suppose that k is algebraically closed if Γ is nontrivial.

(4) *Then we have:*

$$\begin{aligned} S^3(x) &\subset T(x) \cap S_\Gamma(x) \\ T(x) &\subset S^4(x) \text{ if } Z = \mathfrak{Z}(A)^\Gamma, \\ \text{wt}^{-1}(\Gamma(S_A^4(\lambda))) &\supset \bigcup_{\text{wt}(x)=\lambda} S^4(x). \end{aligned}$$

(5) *Condition (S4) holds for $A \rtimes \Gamma$ if it holds for A . If $\mathfrak{Z}(A \rtimes \Gamma) = Z = \mathfrak{Z}(A)^\Gamma$, then the converse is also true, since*

$$S^4(x) = \text{wt}^{-1}(\Gamma(S_A^4(\lambda_x))) \quad \forall x \in X. \quad (11.3)$$

Moreover, (S4) \Rightarrow (S3) in this case.

Note that one of the parts implies that $\chi_\lambda = \lambda \circ \xi \quad \forall \lambda$, and if k is algebraically closed, then $\chi_x = \chi_{\lambda_x} = \theta_{\lambda_x} \quad \forall x \in X$ (see Theorem 11.2).

Proof.

(1) If we prove: $CC_A(\gamma(\lambda)) \supset \gamma(CC_A(\lambda)) \quad \forall \gamma, \lambda$, then

$$\gamma(CC_A(\lambda)) = \gamma(CC_A(\gamma^{-1}(\gamma(\lambda)))) \supset \gamma(\gamma^{-1}(CC_A(\gamma(\lambda)))) = CC_A(\gamma(\lambda)),$$

thereby proving the reverse inclusion. To show the original inclusion, use Proposition 11.1. If $\mu \in CC_A(\lambda)$, then

$$\chi_{\gamma(\mu)}(z) = \gamma\chi_\mu(z)\gamma^{-1} = \gamma\chi_\lambda(z)\gamma^{-1} = \chi_{\gamma(\lambda)}(z)$$

for any central $z \in Z$, whence $\gamma(\mu) \in CC_A(\gamma(\lambda))$.

(2) First, $z \in A$ commutes with A and with Γ if and only if $z \in \mathfrak{Z}(A)^\Gamma$. Next, we prove both inclusions: if $\mu \in S_A^4(\lambda)$ and $\gamma \in \Gamma$, then $\chi_{\gamma(\mu)} \equiv \chi_\mu \equiv \chi_\lambda$ on $\mathfrak{Z}(A)^\Gamma$.

For the reverse inclusion, take any $\mu \notin \Gamma(CC_A(\lambda))$; we will show that $\chi_\mu \neq \chi_\lambda$ on $\mathfrak{Z}(A)^\Gamma$. Since Γ is finite, enumerate the *distinct central characters for A* associated to $\Gamma(CC_A(\mu)) \coprod \Gamma(CC_A(\lambda))$ (equivalently from above, $\Gamma(\lambda) \coprod \Gamma(\mu)$), as $\{\chi_\mu, \chi_1, \dots, \chi_l\}$. (In particular, $\chi_\lambda = \chi_i$ for some i , and so is $\chi_{\gamma(\mu)}$ for all $\gamma(\mu) \notin CC_A(\mu)$.)

Now, $\ker \chi_\mu$ is a maximal ideal in $\mathfrak{Z}(A)$, different from each $\ker \chi_i$; this allows us to choose $z_i \in \mathfrak{Z}(A)$ such that $\chi_i(z_i) = 0 \neq \chi_\mu(z_i)$. Hence $z' = \prod_i z_i$ satisfies: $\chi_i(z') = 0 \quad \forall i$, $\chi_\mu(z') \neq 0$.

Finally, define $z := \sum_{\gamma \in \Gamma} \gamma(z') \in \mathfrak{Z}(A)^\Gamma$. Then if $\nu \in G$,

$$\chi_\nu(z) = \sum_\gamma \chi_\nu(\gamma(z')) = \sum_\gamma \nu(\xi(\gamma(z'))) = \sum_\gamma \gamma^{-1}(\nu(\xi(z'))) = \sum_\gamma \chi_{\gamma(\nu)}(z'),$$

so that from above, $\chi_\lambda(z) = \sum_\gamma \chi_{\gamma(\lambda)}(z') = 0$. On the other hand, $\chi_\mu(z)$ is a sum of zeroes and some (positive) number of $\chi_\mu(z')$'s, which is nonzero (since $\text{char}(k) = 0$ if $|\Gamma| > 1$). Hence $\chi_\mu \neq \chi_\lambda$ on $\mathfrak{Z}(A)^\Gamma$ if $\mu \notin \Gamma(CC_A(\lambda))$, as claimed.

- (3) The second equality comes from Lemma 11.4. We now prove both inclusions for the first equality. Clearly, $\mathfrak{Z}(A)^\Gamma \subset Z$, and conversely, let us claim

Claim. $Z \subset A$.

(If this holds, then $Z = \mathfrak{Z}(A)^\Gamma$ by the previous part.) It remains to show the claim. Suppose $z = \sum_{\gamma \in \Gamma} z_\gamma \gamma \in Z$, with $z_\gamma \in A \forall \gamma$. Given $\gamma \neq 1$, we have to show that $z_\gamma = 0$; now choose any $\lambda \in \mathbb{Z}\Delta$ with $\gamma(\lambda) \neq \lambda$ and $A_\lambda \neq 0$, and fix $0 \neq a_\lambda \in A_\lambda$. Then

$$\sum_{\gamma} a_\lambda z_\gamma \gamma = a_\lambda z = z a_\lambda = \sum_{\gamma} z_\gamma \gamma(a_\lambda) \gamma.$$

We now assume $z_\gamma \neq 0$, and obtain a contradiction. Assume that $z_\gamma = a_{\theta_1} \oplus \cdots \oplus a_{\theta_l}$ for some weight vectors $0 \neq a_{\theta_j} \in A_{\theta_j}$ (with pairwise distinct weights $\theta_j \in G$). Then $a_\lambda a_{\theta_j}$ is nonzero (since A is an integral domain) and in $A_{\lambda+\pi(\theta_j)}$, where all (restricted) weights are in the *abelian* group $\mathbb{Z}\Delta$, by the RTA axioms. Similarly, $0 \neq a_{\theta_j} \gamma(a_\lambda) \in A_{\gamma(\lambda)+\pi(\theta_j)}$.

So if $a_\lambda z = z a_\lambda$, then comparing the coefficient of γ on both sides, the sets of weights on both sides must be the same, whence their sum is the same. Hence $\sum_j \pi(\theta_j) + l\lambda = \sum_j \pi(\theta_j) + l\gamma(\lambda)$, or $l\lambda = l\gamma(\lambda)$, whence (in $\mathbb{Z}\Delta$) $\lambda = \gamma(\lambda)$, a contradiction.

- (4) We first show that if $[Z(x) : V(x')] > 0$, then x and x' are linked. But this is clear, since $Z(x)$ is indecomposable: choose some $N \subset M \subset Z(x)$ such that M is indecomposable, and $M/N \cong V(x')$. (This is possible by Lemma 10.1.) We now have $V(x') \leftarrow M \hookrightarrow Z(x) \twoheadrightarrow V(x)$, so x and x' are linked. This proves that $S^3(x) \subset T(x)$.

Next, that $S^3(x) \subset S_\Gamma(x)$, follows from Proposition 10.2.

It thus remains to show that $T(x) \subset S^4(x)$. To see this, use Theorem 11.2 again: on $Z \subset A$, $\chi_x \equiv \chi_{\lambda_x} \equiv \chi_{\lambda_{F(x)}} \equiv \chi_{F(x)}$, so $F(x) \in S^4(x) \forall x$. Otherwise if x, x' are linked through a chain of indecomposable objects in \mathcal{O} , then there exists a unique “central character block” $\mathcal{O}(\nu)$, that contains all these objects. In particular, since $z - \nu(z)$ kills all simple objects in such a block, we are done.

Finally, we prove the third inclusion. Given $x' \in S^4(x)$ and $\lambda = \lambda_x$, we note that $\chi_x \equiv \chi_{x'}$ on $\mathfrak{Z}(A \rtimes \Gamma) = Z$. Hence on $Z \cap A$, using Theorem 11.2, we get: $\chi_{\lambda_x} \equiv \chi_x \equiv \chi_{x'} \equiv \chi_{\lambda_{x'}}$, whence by a previous part, $\lambda_{x'} \in \Gamma(S_A^4(\lambda_x))$, as desired.

- (5) If $Z \subset A$, then by the previous part, $S^3(x) \subset T(x) \subset S^4(x)$. Next, if A satisfies (S4), then so does $A \rtimes \Gamma$ by the previous part again, using Theorem 5.1. We now only need to prove that if $Z \subset A$, then equation (11.3) holds. But if $\mu \in S_A^4(\lambda_x)$ and $x' \in \text{wt}^{-1}(\mu)$, then $\chi_{\gamma(\mu)} \equiv \chi_\mu \equiv \chi_\lambda$ on $\mathfrak{Z}(A)^\Gamma$. Now use Theorem 11.2; thus, $\chi_{x'} \equiv \chi_x$ on Z - so $x' \in S^4(x)$.

□

11.4. Central characters for wreath products. Finally, we come to the case of $R_{\mathfrak{g}} = S_n \wr \mathfrak{U}\mathfrak{g}$ for \mathfrak{g} a complex semisimple Lie algebra. The main result that we prove here helps prove Proposition 4.2.

Theorem 11.3. *Let $A \rtimes \Gamma = (\mathfrak{U}\mathfrak{g})^{\otimes n} \rtimes S_n = R_{\mathfrak{g}}$ (over $k = \mathbb{C}$).*

- (1) *The center of $R_{\mathfrak{g}}$ is $(\mathfrak{Z}(\mathfrak{U}\mathfrak{g})^{\otimes n})^{S_n} \cong \mathbb{C}[X_1, \dots, X_{ns}]$, with $s = \dim_{\mathbb{C}} \mathfrak{h}$.*
- (2) *The center acts by the same central character on two simple objects $V(x), V(x') \in \mathcal{O}$, if and only if $\lambda_x \in S_n(W_{\mathfrak{g}}^n \bullet \lambda_{x'})$.*
- (3) *The sets of central characters of A and $A \rtimes \Gamma$ are in bijection with $(\mathfrak{h}^*)^n / (W^n, \bullet)$ and $(\mathfrak{h}^*)^{\oplus n} / (S_n \wr W, \bullet)$ respectively. Every central character comes from a Verma module.*

In particular, we obtain a central character block decomposition, by Theorem 11.2 above. The rest of this section is devoted to proving the theorem.

Definition 11.3. Given $\lambda \in G$, define $CC_A(\lambda) := S_A^4(\lambda) = \{\mu \in G : \mu \circ \xi = \lambda \circ \xi \text{ on } \mathfrak{Z}(A)\}$.

Lemma 11.4. *We work over any fixed ground field k .*

- (1) *Suppose a finite group Γ acts by algebra automorphisms on a k -algebra A (here, $|\Gamma| \in k^\times$). Then the fixed point algebra A^Γ is isomorphic to the spherical subalgebra $e_\Gamma A e_\Gamma$ as subalgebras of $A \rtimes \Gamma$, where $e_\Gamma := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma$.*
- (2) *If A_1, \dots, A_n are k -algebras, then $\mathfrak{Z}(\otimes_i A_i) = \otimes_i \mathfrak{Z}(A_i)$.*
- (3) *If A_1, \dots, A_n are RTAs, and $\lambda_i \in G_i$ are weights, then $\chi_\lambda = \otimes_i \chi_{\lambda_i}$ on $\mathfrak{Z}(A)$, and $CC_A(\lambda) = \times_i CC_{A_i}(\lambda_i)$, where $\lambda = (\lambda_1, \dots, \lambda_n)$.*

Proof.

- (1) The map $: A^\Gamma \rightarrow e_\Gamma A e_\Gamma$ ($a \mapsto e_\Gamma a e_\Gamma$) is a k -algebra isomorphism.
- (2) One inclusion is clear; the proof of the other is by induction on n , and the only (possibly) nontrivial step is to show it for $n = 2$. Given k -algebras A, B , suppose $z = \sum_i a_i \otimes b_i$ is central, with the b_i 's linearly independent in B . Choose any $a \in A$; then $az = za$ implies that all a_i 's are central, whence $z \in \mathfrak{Z}(A) \otimes B$. Now write $z = \sum_j a'_j \otimes b'_j$, where the a'_j 's are now linearly independent in $\mathfrak{Z}(A)$. Then $bz = zb$ for all $b \in B$ implies that the b'_j 's are all central in B .
- (3) The statements make sense because of the previous part. It is easy to show that the Harish-Chandra maps ξ_i on A_i and ξ on A satisfy: $\xi = \otimes_i \xi_i$ (extended by linearity). The first part now follows.

One inclusion for the second part follows from the first part here; for the other, by the previous part of this lemma, it suffices to start with $\prod_{i=1}^n \chi_{\lambda_i}(z_i) = \prod_{i=1}^n \chi_{\mu_i}(z_i)$ for all $i, z_i \in \mathfrak{Z}(A_i)$. Now fix i and set $z_j = 1$ for all $j \neq i$; thus $\mu_i \in CC_{A_i}(\lambda_i) \forall i$.

□

It is not hard now, to compute the center of $R_{\mathfrak{g}} = S_n \wr \mathfrak{U}\mathfrak{g}$, or the corresponding blocks. We can now prove the main result in this subsection.

Proof of Theorem 11.3.

- (1) The first claim follows from Lemma 11.4, Proposition 11.3, and the following two classical results from Lie theory ($s = \dim_{\mathbb{C}} \mathfrak{h}$ here):
 - (a) $\mathfrak{Z}(\mathfrak{U}\mathfrak{g}) \cong \mathbb{C}[X_1, \dots, X_s]$ (see [Dix, Theorem 7.3.8]).
 - (b) If G is a finite group acting on a finitely generated polynomial algebra $\mathbb{C}[X_1, \dots, X_l]$ via reflections, then $\mathbb{C}[X_1, \dots, X_l]^G \cong \mathbb{C}[Y_1, \dots, Y_l]$ (Chevalley's Theorem; see [Che]).
 Applying these results (with $G = S_n$), the first part follows.
- (2) By Harish-Chandra's Theorem, $(CC_{\mathfrak{U}\mathfrak{g}}(\lambda) = W_{\mathfrak{g}} \bullet \lambda$. Now by Lemma 11.4,) $CC_{(\mathfrak{U}\mathfrak{g})^{\otimes n}}(\lambda_1, \dots, \lambda_n) = W_{\mathfrak{g}}^n \bullet (\lambda_1, \dots, \lambda_n)$. We are now done by Proposition 11.3 (and the previous part).
- (3) The only part not done above, involves computing *all* central characters of $A \rtimes \Gamma$ (and not merely those in the Category \mathcal{O}) - note that the result itself implies that every central character corresponds to some object in \mathcal{O} . This remaining part follows from a special case of the Nagata-Mumford Theorem (see e.g., [Muk, Theorem 5.3]).

□

Finally, we present the proof of an earlier, unproved result:

Proof of Proposition 4.2. All but the first part (and that \mathcal{O} is finite length) were shown in this section. However, by Harish-Chandra's theorem, $\mathfrak{U}\mathfrak{g}$ satisfies all the Conditions (S). Thus, we are done by Theorem 4.2. Moreover, that $\mathfrak{U}\mathfrak{g}^{\otimes n} = \mathfrak{U}(\mathfrak{g}^{\oplus n})$ is of finite representation type, is well-known.

Now consider a finite-dimensional simple $R_{\mathfrak{g}}$ -module; it is also a finite-dimensional $\mathfrak{U}\mathfrak{g}^{\otimes n}$ -module, hence is in $\mathcal{O}_{R_{\mathfrak{g}}}$. So let us denote it by $V(x)$; say $\dim V(x) = d$. Now by Corollary 7.1, $\dim V_A(\lambda_x) | d$, and there are only finitely many such λ_x 's. We are now done by Theorem 5.1. □

12. EACH BLOCK IS A HIGHEST WEIGHT CATEGORY

We now show that each block $\mathcal{O}(x)$ has enough projectives, and is a highest weight category, under some Condition (S). The following result (see [Don, (A1)]) will be useful shortly; the proof is similar to [Kh1, Theorem 3].

Proposition 12.1.

- (1) If $\text{Ext}_{\mathcal{O}}^1(Z(x), M)$ or $\text{Ext}_{\mathcal{O}}^1(M, F(Z(x)))$ is nonzero for $M \in \mathcal{O}$ and $x \in X$, then M has a composition factor $V(x')$ with $x' > x$.
- (2) If X and $F(Y)$ have simple Verma flags, then $\text{Ext}_{\mathcal{O}}^1(X, Y) = 0$, and

$$\dim_k(\text{Hom}_A(X, Y)) = \sum_{x \in X} [X : Z(x)][Y : F(Z(F(x)))] \dim_k \text{End}_{A \rtimes \Gamma} V(x).$$

Given $x \in X$, we now define $\mathcal{O}^{\leq x}$ (or $\mathcal{O}^{< x}$) to be the subcategory of objects $N \in \mathcal{O}$, so that all simple subquotients of N are of the form $V(x')$ for $x' \leq x$ (or $x' < x$ respectively). Given $x, x' \in X$, define $\mathcal{O}(x')^{\leq x} := \mathcal{O}(x') \cap \mathcal{O}^{\leq x}$, and similarly, $\mathcal{O}(x')^{< x}$ - so $Z(x) \in \mathcal{O}(x)^{\leq x}$ and $Y(x) \in \mathcal{O}(x)^{< x}$.

We first show that enough projectives exist in \mathcal{O} .

Lemma 12.1. *If $A \rtimes \Gamma$ satisfies Condition (S1), then $Z(x)$ is the projective cover of $V(x)$ in $\mathcal{O}(x)^{\leq x}$.*

Proof. We know that $\mathcal{O}(x)^{\leq x} \subset \mathcal{O}(\lambda_x, 1)$, so by Proposition 9.1, $Z(x) = P(x, 1)$ is projective here. Moreover, $Z(x)$ is indecomposable, with radical $Y(x)$. The usual Fitting Lemma arguments now complete the proof. \square

Proposition 12.2. *If $A \rtimes \Gamma$ satisfies Condition (S2), then each $\mathcal{O}(x)$ has enough projectives. If $A \rtimes \Gamma$ satisfies Condition (S3), then each block $\mathcal{O}(x)$ is equivalent to the category $(\text{Mod } -B)^{fg}$ of finitely generated right modules over a finite-dimensional k -algebra $B = B_x$.*

Proof. The proof of the first part uses the x -component in the (block) decomposition of some $P(y, l)$, as in [BGG]. The second part uses the existence of progenerators and the usual Fitting lemma arguments, as in [Kh1]. \square

Let us denote the projective cover of $V(x)$ by $P(x)$.

Proposition 12.3. *For all $x \in X$ and $M \in \mathcal{O}$, we have*

$$\begin{aligned} \text{Hom}_{\mathcal{O}}(P(x), V(x')) &= \delta_{x, x'} \text{End}_{\mathcal{O}} V(x), \\ \dim_k(\text{Hom}_{\mathcal{O}}(P(x), M)) &= [M : V(x)] \dim_k \text{End}_{\mathcal{O}} V(x). \end{aligned}$$

Proof. Both sides of the second equation are additive in M , over short exact sequences. This reduces it to the case $M = V(x')$, i.e., the previous equation, which holds by general properties of projective covers. \square

To show that each block $\mathcal{O}(x)$ is a highest weight category (see [CPS] for the definition), we need a result from [BGG]; its proof is also valid here.

Proposition 12.4. *Recall what a p -filtration means, in Definition 6.1 above.*

- (1) *If $M \in \mathcal{O}$ has a p -filtration, and $x \in X$ is maximal (minimal) in the set of Verma subquotients $Z(x)$ of M , then M has a submodule (quotient) $Z(x)$, and the quotient (kernel of the quotient, respectively) has a simple Verma flag.*
- (2) *Given $M_1, M_2 \in \mathcal{O}$, $M = M_1 \oplus M_2$ has a p -filtration if and only if each of M_1 and M_2 has a simple Verma flag.*

Corollary 12.1. *If $A \rtimes \Gamma$ satisfies Condition (S2), then every $P(x)$ has a p -filtration, with one subquotient (the “first” one) $Z(x)$, and all others $Z(x')$ for some $x' > x$.*

Proof. This is because each $P(x, l)$ (see Proposition 9.1) has a simple Verma flag, with one subquotient $Z(x)$, and every other subquotient $Z(x')$ for some

$x' \geq x$. Now use Propositions 12.4, 9.1, and 12.3, as in [Kh1, Theorem 6]. That $Z(x)$ is the “highest” subquotient is as in Proposition 12.4 (or from [Don, (A3.1)(i)]). \square

Theorem 12.1. *Each block $\mathcal{O}(x)$ is a highest weight category if $A \rtimes \Gamma$ satisfies Condition (S3).*

Proof. We need Condition (S3) to ensure that the set of simple objects is “interval-finite” with respect to the partial order. We are now done by Corollary 12.1. \square

13. BGG RECIPROCITY AND THE (SYMMETRIC) CARTAN MATRIX

Standing Assumption 13.1. For this section, $A \rtimes \Gamma$ satisfies Condition (S3), and if Γ is nontrivial, we require that k is algebraically closed.

Definition 13.1. Fix a block $\mathcal{O}(x)$, and order $S^3(x)$ such that $x_i \geq x_j \Rightarrow i \leq j$. We then define the

- *decomposition matrix* D_x by: $(D_x)_{ij} := [Z(x_i) : V(x_j)]$.
- *duality matrix* F_x by: $(F_x)_{ij} := \delta_{x_i, F(x_j)}$.
- *Cartan matrix* C_x by: $(C_x)_{ij} := [P(x_i) : V(x_j)]$.
- *modified Cartan matrix* C'_x by: $(C'_x)_{ij} := [P(x_i) : V(F(x_j))]$.

Note that the duality functor F does not preserve each $V(x)$, so the “usual” notion of the Cartan matrix is not symmetric, as in the classical case of \mathfrak{Ug} . However, a variant is.

Proposition 13.1 (BGG Reciprocity). *For all $x, x' \in X$, we have*

$$[P(x') : Z(x)] = [Z(F(x)) : V(F(x'))] \quad (13.1)$$

and hence C'_x is symmetric; more precisely, $C'_x = F_x D_x^T F_x D_x F_x$.

As a consequence, [GK, Theorem 9.1] holds, reconciling various notions of block decomposition; in particular, $S^3(x) = T(x)$ (recall Proposition 11.3).

Proof. Proposition 8.3 implies that $V(x)$ is Schurian now, for all x . Hence using Propositions 12.1 and 12.3, we get that

$$\begin{aligned} [P(x') : Z(x)] &= \dim_k \operatorname{Hom}_{\mathcal{O}}(P(x'), F(Z(F(x)))) = [F(Z(F(x))) : V(x')] \\ &= [Z(F(x)) : F(V(x'))] = [Z(F(x)) : V(F(x'))]. \end{aligned}$$

The second part is also standard, say, from the following results. \square

Proposition 13.2. *Let $\operatorname{Grot}_{\mathcal{D}}$ be the Grothendieck group of a category \mathcal{D} .*

- (1) $\operatorname{Grot}_{\mathcal{O}} = \oplus \operatorname{Grot}_{\mathcal{O}(x)}$.
- (2) F_x is symmetric and has order at most two.
- (3) $C_x = F_x D_x^T F_x D_x$. In particular, if $\Gamma = 1$ then C_x is symmetric.
- (4) Each of the following sets is a \mathbb{Z} -basis for $\operatorname{Grot}_{\mathcal{O}}$:

$$\{[V(x)] : x \in X\}, \{[Z(x)] : x \in X\}, \{[P(x)] : x \in X\}.$$

Proof. The first and last part are standard in a highest weight category; the second part holds because $F(F(V(x))) = V(x) \forall x$; and the third part follows from equation (13.1) above. \square

14. THE SECOND SETUP - TENSOR PRODUCTS

We now look at the representation theory of tensor products of skew group rings over regular triangular algebras. More precisely, we relate the category \mathcal{O} over such a product, to the respective categories \mathcal{O}_i over each factor.

14.1. Notation. Fix $n \in \mathbb{N}$. We fix skew group rings $A_i \rtimes \Gamma_i$ over RTAs A_i , that satisfy the standing assumptions 2.1 and 6.1. Then (mentioned above) so does $A \rtimes \Gamma := \otimes_{i=1}^n (A_i \rtimes \Gamma_i)$, where $A = \otimes_i A_i$, $\Gamma = \times_i \Gamma_i$, and $G = \times_i G_i$.

14.2. Duality and tensor product decomposition. We first mention some exact functors on tensor products of \mathcal{O}_j 's. For this subsection, we work with the setup mentioned in Section 8 (on duality) above: we have associative k -algebras A'_j , each containing a unital k -subalgebra H'_j ; moreover, there exist anti-involutions i_j of each A'_j , that extend $\text{id}_{H'_j}$.

We can now define $A' = \otimes_j A'_j$, and similarly, H', i , and the Harish-Chandra categories \mathcal{H}'_j and $\mathcal{H}' \supset \otimes_j \mathcal{H}'_j$. Objects in these categories all have “formal characters”, and the (restricted) duality functor F operates on each of these categories. The following is now standard.

Proposition 14.1. *Fix $V_j \in \mathcal{H}'_j$ for all j , and fix $1 \leq i \leq n$.*

- (1) $F(\otimes_j V_j) \cong \otimes_j F(V_j)$, and $\text{ch}_{\otimes_j V_j} = \prod_j \text{ch}_{V_j}$.
- (2) The functor $T_{\mathbf{V}} : \mathcal{H}'_i \rightarrow \mathcal{H}'$, sending an A'_i -module M to the A' -module $T_{\mathbf{V}}(M) := (\otimes_{j < i} V_j) \otimes M \otimes (\otimes_{j > i} V_j)$, is exact.
- (3) As A'_i -modules, $T_{\mathbf{V}}(M)$ is a direct sum of copies of M ; more precisely, $T_{\mathbf{V}}(M) \cong M \otimes W$, for the vector space $W = \otimes_{j \neq i} V_j$.

We now apply this in our setup. Define the Harish-Chandra and BGG Categories \mathcal{H} (or \mathcal{H}_i) and \mathcal{O} (or \mathcal{O}_i) respectively, for $A \rtimes \Gamma \supset H$ (or $A_i \rtimes \Gamma_i \supset H_i$ respectively). Then the above result holds.

Corollary 14.1. *If $\otimes_i V_i(x_i) \cong \otimes_i V_i(x'_i)$ for $x_i, x'_i \in X_i$ for all i , then $x_i = x'_i \forall i$. Moreover, if $\mathbf{y} = \otimes_i y_i$, then $Z(\mathbf{y}) \cong \otimes_i Z_i(y_i)$.*

Proof. For the first part, apply Proposition 14.1 to both sides (having first fixed an i). Thus, the left side is a direct sum of copies of $V_i(x_i)$, and similarly for the other side. Now apply Lemma 10.1 (for $R = A_i \rtimes \Gamma_i$); thus $V_i(x_i) \cong V_i(x'_i)$ is the only simple summand on both sides, and we are done.

For the second part, we note that $\times_i Y_i \subset Y$, so if we define \mathbf{y} as above, then $Z(\mathbf{y}) \rightarrow \otimes_i Z_i(y_i) \rightarrow 0$. Moreover, properties of induction functors (and the triangular decomposition) imply that we can compare their formal characters:

$$\text{ch}_{Z(\mathbf{y})} = \text{ch}_{B_-} \text{ch}_{\mathbf{y}} = \text{ch}_{\otimes_i B_{i,-}} \text{ch}_{\otimes_i y_i} = \prod_i \text{ch}_{B_{i,-}} \text{ch}_{y_i} = \prod_i \text{ch}_{Z_i(y_i)},$$

whence the two must be isomorphic. \square

15. COMPLETE REDUCIBILITY

We now show that all notions of complete reducibility (i.e., in the four setups at the start, and always only for finite-dimensional objects in \mathcal{O}) are equivalent. We need a small result first, since two of the parts below are similar. For this section, we do not need k to be algebraically closed.

Proposition 15.1. *Suppose A' and A are RTAs, and $A \rtimes \Gamma$ is a skew group ring satisfying the Standing Assumptions 2.1 and 6.1. Also say there is a finite-dimensional vector space U and $0 \neq u \in U$, such that*

- (1) $T : M \mapsto U \otimes_k M$ is an exact covariant functor $: \mathcal{O}_{A'} \rightarrow \mathcal{O}_{A \rtimes \Gamma}$.
- (2) If $v_{\mu'}$ has highest weight in the A' -Verma module $Z_{A'}(\mu')$, then $v_{\mu'} \otimes u$ has highest weight in $T(Z_{A'}(\mu'))$, which is also standard cyclic.
- (3) The map: $-\otimes u$ takes weight spaces to weight spaces, and the induced map $: G_{A'} \rightarrow G_A$ is compatible with the partial orders on $G_{A'}, G_A$.

Then if complete reducibility holds for finite-dimensional modules in $\mathcal{O}_{A \rtimes \Gamma}$, the same holds in $\mathcal{O}_{A'}$.

Proof. We will show that every short exact sequence between simple finite-dimensional objects $0 \rightarrow V(\lambda') \rightarrow V \rightarrow V(\mu') \rightarrow 0$ in $\mathcal{O}_{A'}$ splits. Now assume that there is some such nonsplit sequence. We may assume (using Proposition 9.3 with $|\Gamma| = 1$, and) using the duality functor F if necessary, that $\mu' > \lambda'$. Then V is standard cyclic; say $v_{\mu'}$ spans its μ' -weight space.

Now apply T to the sequence; by assumption, we get a short exact sequence in $\mathcal{O}_{A \rtimes \Gamma}$, and each object is finite-dimensional since U is. On the one hand, $T(V)$ is standard cyclic, and generated by $v_{\mu'} \otimes u$ (by assumption). On the other hand, the short exact sequence in $\mathcal{O}_{A \rtimes \Gamma}$ splits, and by assumption, $v_{\mu'} \otimes u$ has higher weight than the weights for $T(V(\lambda'))$ - whence it must lie in the complement to $T(V(\lambda'))$. In particular, it cannot generate the entire module $T(V)$, a contradiction. \square

We now prove the equivalence of complete reducibility in the four setups (we do this in two stages). We assume that all these setups involve (skew group rings over) RTAs, satisfying Standing Assumptions 2.1 and 6.1, but not necessarily any of the Conditions (S).

Theorem 15.1. *Given such a skew group ring $A \rtimes \Gamma$, complete reducibility holds (for finite-dimensional modules) in $\mathcal{O}_A \subset A\text{-mod}$, if and only if it holds in $\mathcal{O}_{A \rtimes \Gamma}$.*

Proof. First note that the existence of finite-dimensional (simple) modules in \mathcal{O}_A and in $\mathcal{O}_{A \rtimes \Gamma}$ is equivalent, by Theorems 5.1 and 7.1. Now suppose that complete reducibility holds in \mathcal{O}_A . Apply Proposition 3.1 (using Lemma 3.1 above), to the abelian subcategories \mathcal{P}, \mathcal{D} of H -semisimple finite-dimensional modules inside $\mathcal{O}_A, \mathcal{O}_{A \rtimes \Gamma}$ respectively. This proves one implication.

Conversely, apply Proposition 15.1, with $A' = A, U = k\Gamma, u = 1 \in U$. (Then $T = \text{Ind}_A^{A \rtimes \Gamma}$ is exact, and the other assumptions also hold; for instance, $T(V_A(\lambda)) = V(y)$, where $y = \text{Ind}_H^{H \rtimes \Gamma} k^\lambda$ for $\lambda \in G$.) \square

Now suppose we are working with skew group rings (as above) $A_i \rtimes \Gamma_i$, and we define $A := \otimes_i A_i, \Gamma := \times_i \Gamma_i$.

Theorem 15.2. *Complete reducibility holds in \mathcal{O} if and only if it does so in all \mathcal{O}_i .*

Proof. We point out that we will use previous results, as well as Proposition 16.2 (below) in the setting $|\Gamma| = 1$.

We now show the result. First, by Theorem 15.1, it is enough to check this for $|\Gamma| = 1$. Hence $\mathcal{O}_i = \mathcal{O}_{A_i}$ and $\mathcal{O} = \mathcal{O}_A$. Next, the existence of finite-dimensional modules in \mathcal{O} is equivalent to that in \mathcal{O}_i for all i ; this follows from Proposition 16.2 (when $|\Gamma| = 1$).

Therefore we now assume that there exist finite-dimensional modules in each \mathcal{O}_i (and in \mathcal{O}). Suppose complete reducibility holds in each \mathcal{O}_i , and we have a non-split short exact sequence $0 \rightarrow V(\lambda) \rightarrow V \rightarrow V(\mu) \rightarrow 0$ of finite-dimensional modules (i.e., an indecomposable module of length 2). From Proposition 9.3, $\lambda > \mu$ or $\lambda < \mu$ if the sequence does not split; using the duality functor F if necessary, assume that $\lambda < \mu$.

Fix an i such that $\mu_i > \lambda_i$; now by assumption, the sequence does split as finite-dimensional A_i -modules. Suppose $V = V(\lambda) \oplus M$ in \mathcal{O}_i . By Proposition 9.3, V is a standard cyclic A -module; say $V = Av_\mu$. Then by H_i -semisimplicity and Proposition 14.1, v_μ is in the H_i -weight space $V_{\mu_i} = V(\lambda)_{\mu_i} \oplus M_{\mu_i} = M_{\mu_i}$ (since $\lambda_i < \mu_i$, and using Proposition 16.2 below, for $|\Gamma| = 1$). But $M \cong V(\mu)$ is a direct sum of copies of $V_i(\mu_i)$. Hence so is $A_i v_\mu \subset M$, and hence also, $V = Av_\mu = (\otimes_{j \neq i} A_j) \cdot A_i v$ - hence, its A_i -submodule $V(\lambda) \cong \bigoplus V_i(\lambda_i)$ (as A_i -modules) as well - and we get a contradiction by Lemma 10.1.

Hence all extensions with $\lambda < \mu$ split, and by duality, so do all extensions with $\lambda > \mu$. Thus complete reducibility holds in \mathcal{O} .

Conversely, fix i , and for all $j \neq i$, fix simple finite-dimensional modules $V_j(\lambda_j) \in \mathcal{O}_j$ (these exist by above remarks). Define the functor $T_{\mathbf{V}} : \mathcal{O}_i \rightarrow \mathcal{O}$ as in Proposition 14.1 above; thus, $T_{\mathbf{V}}$ is exact.

Now apply Proposition 15.1 with $A' = A_i, |\Gamma| = 1, A$ as above, $U = \otimes_{j \neq i} V_j(\lambda_j)$ (so $T = T_{\mathbf{V}}$), and $u = \otimes_{j \neq i} v_{j, \lambda_j}$, the unique (up to scalars) highest weight vector in U . (Note that $T_{\mathbf{V}}(V_i(\lambda_i)) = V_A(\lambda_1, \dots, \lambda_n)$ by Proposition 16.2 again.) \square

16. CONDITION (S) FOR TENSOR PRODUCTS

The idea now, is to relate the Categories \mathcal{O}_i for $A_i \rtimes \Gamma_i$, to $\mathcal{O} = \mathcal{O}_{A \rtimes \Gamma}$. We need to characterize the simple objects in the latter, in terms of those in

the former categories. As above, we have the various sets X_i (X) of simple $H_i \rtimes \Gamma_i$ - ($H \rtimes \Gamma$ -) modules that are H_i - (respectively H -) semisimple.

Standing Assumption 16.1. For this section and the next, assume that Standing Assumption 4.1 holds.

Theorem 16.1. $X = \times_j X_j$.

The example one should have in mind, is the case $|\Gamma| = 1$; then $X_j = G_j = \text{Hom}_{k\text{-alg}}(H_j, k)$, and $X = G = \times_j G_j$. Also note that if $\mathbf{x} \in X$, then $\lambda_{\mathbf{x}} = (\lambda_{x_1}, \dots, \lambda_{x_n}) \in \times_j G_j$.

Proof. For this, we need the following “general” result¹.

Proposition 16.1. *Let k be an algebraically closed field; all tensor products are over k . Let R_i be unital k -algebras for $1 \leq i \leq n$, and define $R := \otimes_i R_i$. If P_i (or P) is the set of (isomorphism classes of) finite-dimensional simple modules over R_i (or R , respectively) (for each i), then the map $\otimes : \times_i P_i \rightarrow R\text{-Mod}$, taking $([M_1], \dots, [M_n])$ to $[\otimes_i M_i]$, is a bijection onto P .*

The proof of this result is an exercise in Wedderburn theory, keeping in mind that if a simple A -module M is finite-dimensional (over k , where A is a k -algebra), then A acts on M via a subalgebra $A' \subset \mathfrak{gl}_k(M)$. (So A' is Artinian.) Moreover, M is a faithful simple A' -module - which makes A' a simple algebra, hence of the form $\text{End}_k(k^l)$.

We now prove the theorem. If $|\Gamma| = 1$ then the result is clear, since $G = \times_j G_j$. If not, then k is algebraically closed of characteristic zero, by assumption. By Proposition 16.1, we only need to show that if $M_{\mathbf{x}} := \otimes_i M_{x_i}$, then the M_{x_i} are H_i -semisimple if and only if $M_{\mathbf{x}}$ is H -semisimple. The “only if” part is clear, and for the “if” part, it is not hard to show that for each $\lambda \in G$, $(M_{\mathbf{x}})_{\lambda} \subset \otimes_i (M_{x_i})_{\lambda_i}$. Therefore

$$M_{\mathbf{x}} = \bigoplus_{\lambda \in G = \times_i G_i} (M_{\mathbf{x}})_{\lambda} \subset \bigoplus_{\lambda \in G} \bigotimes_i (M_{x_i})_{\lambda_i} = \bigotimes_i \bigoplus_{\lambda_i \in G_i} (M_{x_i})_{\lambda_i} \subset \otimes_i M_{x_i} = M_{\mathbf{x}}$$

whence all inclusions are equalities, and $M_{x_i} = \bigoplus_{\lambda_i \in G_i} (M_{x_i})_{\lambda_i} \forall i$. \square

It is now easy to determine the simple objects in \mathcal{O} (and their characters in terms of those of the simple objects in \mathcal{O}_i):

Proposition 16.2 (“Weyl Character Formula 2”). *$V \in \mathcal{O}$ is simple if and only if $V = \otimes_i V_i(x_i)$ for some simple $V_i(x_i) \in \mathcal{O}_i$. (Moreover, $\text{ch}_V = \prod_i \text{ch}_{V_i(x_i)}.$)*

Proof. Fix i . Given $x_j \in X_j$ for all j , $V := \otimes_j V_j(x_j)$ is isomorphic to $V_i(x_i) \otimes W$ (as $A_i \rtimes \Gamma_i$ -modules) by Proposition 14.1 - where W is the vector space $\otimes_{j \neq i} V_j(x_j)$. In particular, any maximal vector in V generates an $A_i \rtimes \Gamma_i$ -submodule; this submodule must also be a direct sum of copies of $V_i(x_i)$, whence the vector has i th weight component $\gamma_i(\lambda_i)$ for some $\gamma_i \in \Gamma_i$.

¹I thank Mitya Boyarchenko for telling me this general result.

This holds for every $1 \leq i \leq n$; thus any maximal vector in V has weight $\gamma(\lambda)$, whence it is in $\otimes_i M_{x_i} = M_{\mathbf{x}}$. But $M_{\mathbf{x}}$ is simple by Theorem 16.1, so the standard cyclic module $\otimes_j V_j(x_j)$ is simple, whence it equals $V(\mathbf{x})$.

Thus, the map $\Psi : \times_i X_i \rightarrow X$ (taking $(V_i(x_i))_i$ to $\otimes_i V_i(x_i)$) is surjective. By Corollary 14.1 above, Ψ is also injective; hence we are done. \square

Corollary 16.1. $F(\mathbf{x}) = (F(x_j))_j$ in \mathcal{C} , and $F(V(\mathbf{x})) = \otimes_j F(V_j(x_j))$ in \mathcal{O} .

Proof. The statement in \mathcal{C} follows from Proposition 14.1, and the other equation now follows by using Propositions 16.2 and 8.2. \square

We now relate Condition (S3) for $A \rtimes \Gamma$, with the same condition for all $A_i \rtimes \Gamma_i$. Define the sets $S'_i(x_i) \subset S^3_i(x_i) \subset X_i$ (and $S'(\mathbf{x}) \subset S^3(\mathbf{x}) \subset X$) as in Definition 4.1 above.

Theorem 16.2.

- (1) For each \mathbf{x} , $Z(\mathbf{x})$ has finite length if and only if each $Z_j(x_j)$ does.
- (2) \mathcal{O} is finite length if and only if all \mathcal{O}_i 's are finite length.
- (3) Given $x_j \in X_j$ for all j , $S'(\mathbf{x}) \subset \times_j S'_j(x_j) \subset S^3(\mathbf{x}) \subset \times_j S^3_j(x_j)$.

Proof. We will need the following elementary result.

Lemma 16.1. Given a ring R_j and an R_j -module M_j for each $1 \leq j \leq n$, define $R := \otimes_j R_j$ and $M := \otimes_j M_j$. If each M_j has a chain of submodules $M_j = M_{j,0} \supsetneq M_{j,1} \supsetneq \cdots \supsetneq M_{j,l_j} = 0$, then M has a chain of length $\prod_j l_j$, with set of subquotients $\{\otimes_j (M_{j,i_j-1}/M_{j,i_j}) : 1 \leq i_j \leq l_j \ \forall j\}$.

- (1) If $l_j = l(Z_j(x_j))$, then $Z(\mathbf{x})$ has length $\prod_j l_j$ by Lemma 16.1 and Proposition 16.2. Conversely, suppose some $Z(\mathbf{x})$ has finite length, say n , but some $Z_i(x_i)$ does not. Then we can construct an arbitrarily long filtration of $Z_i(x_i)$, say of length $> n$. By Lemma 16.1, this gives a filtration of $Z(\mathbf{x})$ of length larger than n , a contradiction.
- (2) This follows from Proposition 9.2 and the previous part.
- (3) This is done in stages.

Step 1. We show the first inclusion (and will use it later, to show the third inclusion). We are to show that if $\mathbf{x}' \in \times_j S'_j(x_j)$, and $[Z(\mathbf{x}') : V(\mathbf{x}'')] \neq 0$ or $[Z(\mathbf{x}'') : V(\mathbf{x}')] \neq 0$, then $\mathbf{x}'' \in \times_j S'_j(x_j)$.

Suppose $[Z(\mathbf{x}') : V(\mathbf{x}'')] > 0$ (the proof is similar in the other case). Fix i . As $A_i \rtimes \Gamma_i$ -modules, $Z(\mathbf{x}') = \otimes_j Z_j(x'_j)$ (or $V(\mathbf{x}'') = \otimes_j V_j(x''_j)$) is isomorphic (by above results) to a direct sum of copies of $Z_i(x'_i)$ (or $V_i(x''_i)$, respectively). Thus, $V_i(x''_i)$ is a subquotient of $V(\mathbf{x}'')$ (as $A_i \rtimes \Gamma_i$ -modules), hence also of $Z(\mathbf{x}') = \oplus Z_i(x'_i)$. Now use Lemma 10.1 - so $x''_i \in S'_i(x'_i) = S'_i(x_i)$ (by transitivity) for all i .

Step 2. Since in a Cartesian product of graphs, the connected components are the products of connected components in each factor, the second inclusion holds if we show that if $\mathbf{x}' \in S^3(\mathbf{x})$ and

$x_i'' \in S'_i(x_i)$ for some i , then $\mathbf{x}'' := (x'_1, \dots, x'_{i-1}, x_i'', x'_{i+1}, \dots, x'_n)$ is in $S^3(\mathbf{x})$. By transitivity, this further reduces to showing the same when $[Z_i(x'_i) : V_i(x_i'')]$ or $[Z_i(x_i'') : V_i(x'_i)]$ is nonzero.

We show the proof in the second case (the proof in the first case is similar). Apply Lemma 16.1 to the chains of submodules

$$M_j = Z_j(x'_j) \supsetneq Y_j(x'_j) \supset 0 \quad \forall j \neq i, \quad M_i = Z_i(x_i'') \supset M' \supsetneq N' \supset 0,$$

where $M'/N' \cong V_i(x'_i)$. Then $[Z(\mathbf{x}'') : V(\mathbf{x}')] > 0$, whence $\mathbf{x}'' \in S^3(\mathbf{x}') = S^3(\mathbf{x})$ (by transitivity).

Step 3. Finally, the third inclusion follows from Step 1 and Corollary 16.1, since $\times_j S_j^3(x_j)$ is now closed under both operations. \square

Thus, the last part is a step towards showing the equivalence of Condition (S3) in the two setups. In fact, it is enough in the case that we need:

Corollary 16.2. *If $|\Gamma| = 1$, then $S_A^3(\lambda_1, \dots, \lambda_n) = \times_j S_j^3(\lambda_j) \quad \forall i, \lambda_i \in G_i$; hence Condition (S3) holds for A if and only if it holds for every A_i .*

Proof. If $|\Gamma| = 1$, $X = G$, $X_i = G_i \quad \forall i$, and $S'(\lambda) = S^3(\lambda) \quad \forall \lambda$ (since $F(\lambda) = \lambda$); thus, all inclusions in the final part of the above result are equalities. \square

We conclude this section by simultaneously doing two things. First, the last part of Theorem 16.2 suggests that some generalization of the formula in Corollary 16.2 is possible. Moreover, we wish to complete the following “commuting” cube (relating the “ S^3 -sets” in various setups), given $\gamma \in \Gamma$, $\lambda \in G$, and $\mathbf{x} \in X$ such that $\lambda = \lambda_{\mathbf{x}}$ (also see Proposition 10.2); note that F preserves every (known) vertex.

$$\begin{array}{ccccc}
 \{S_i^3(\gamma(\lambda_i))\} & \xrightarrow{\times} & \{S_i^3(x_i)\} & & (16.1) \\
 \nearrow \gamma(\cdot) & & \nearrow \gamma(\cdot)=\text{id} & & \\
 \{S_i^3(\lambda_i)\} & \xrightarrow{\times} & \{S_i^3(x_i)\} & & \\
 \downarrow \times & & \downarrow \times & & \\
 S_A^3(\lambda) & \xrightarrow{\times} & S_A^3(\gamma(\lambda)) & \xrightarrow{\times} & ? \\
 \nearrow \gamma(\cdot) & & \nearrow \gamma(\cdot)=? & & \\
 S_A^3(\lambda) & \xrightarrow{\times} & ? & &
 \end{array}$$

To do this, we introduce the following notation. Define $F^0(x) := x$ and $F^1(x) := F(x)$; for all $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n$, define $F^\varepsilon(\mathbf{x}) := (F^{\varepsilon_1}(x_1), \dots, F^{\varepsilon_n}(x_n))$. For all $\varepsilon, \varepsilon'$, we thus have: $F^\varepsilon(F^{\varepsilon'}(\mathbf{x})) = F^{\varepsilon+\varepsilon'}(\mathbf{x})$.

Now note that the cube above is identical to the one in diagram (4.2) above, with the Z 's replaced by S^3 . However, the missing set of objects $?$ (which should be contained in $\times_i S_i^3(x_i)$) *cannot* just be $S^3(\mathbf{x})$, since it

could also have been $S^3(F^\varepsilon(\mathbf{x}))$ in its place. So the natural candidate now, would be the finite union $\bigcup_\varepsilon S^3(F^\varepsilon(\mathbf{x}))$. The question now is: when is this set closed under all the F^ε ? We present a sufficient condition.

Theorem 16.3. *If $F(S'_i(x_i)) = S'_i(F(x_i)) \ \forall i, x_i$, then*

$$\times_j S_j^3(x_j) = \bigcup_{\varepsilon \in \mathbb{P}(\mathbb{Z}/2\mathbb{Z})^n} S^3(F^\varepsilon(\mathbf{x})). \quad (16.2)$$

Remark 16.1.

- (1) Here, $\mathbb{P}(\mathbb{Z}/2\mathbb{Z})^n$ denotes the quotient of $(\mathbb{Z}/2\mathbb{Z})^n$ by the diagonal copy of $\mathbb{Z}/2\mathbb{Z}$ sitting in it as $\{(0, \dots, 0), (1, \dots, 1)\}$ (we abuse notation). This is because by Corollary 16.1, $F(\mathbf{x}) = F^{(1, \dots, 1)}(\mathbf{x}) \in S^3(\mathbf{x})$.
- (2) In general, it is not clear that $F(S'(x)) = S'(F(x))$. In the degenerate case $\Gamma = 1$, this holds because $F(\lambda) = \lambda$ for any RTA A and any $\lambda \in G$. (So $F(S'(\lambda)) = S'(\lambda) = S^3(\lambda)$.)

Proof. Note that $S^3(F^\varepsilon(\mathbf{x})) \subset \times_j S_j^3(F^{\varepsilon_j}(x_j)) = \times_j S_j^3(x_j) \ \forall \varepsilon$. Moreover, Theorem 16.2 implies that we can cover all of $\times_j S_j^3(x_j)$ from any fixed \mathbf{x} , using the duality functors F^ε , and the relation of being a simple subquotient of a Verma module.

Claim. $F^\varepsilon(S^3(\mathbf{x})) = S^3(F^\varepsilon(\mathbf{x})) \ \forall \varepsilon$.

Before we show the claim, let us use it to prove the theorem. Note, by the first paragraph in this proof, that the right-hand side of equation (16.2) is contained in the left side. Moreover, both sides are closed under the “Verma-to-simple relation”, as well as all possible F^ε ’s (by the claim). But both sides also partition X (since $X_j = \coprod S_j^3(x_j)$ for all j), whence all inclusions are now equalities, as desired.

It remains to prove the claim. Moreover, it suffices to show, for any coordinate vector e_i , that $F^{e_i}(S^3(\mathbf{x})) \subset S^3(F^{e_i}(\mathbf{x}))$, for then we get that

$$S^3(F^{e_i}(\mathbf{x})) = F^{e_i}(F^{e_i}(S^3(F^{e_i}(\mathbf{x})))) \subset F^{e_i}(S^3(\mathbf{x})) \subset S^3(F^{e_i}(\mathbf{x}))$$

from above, whence all inclusions become equalities. The statement for general ε follows by a series of compositions of various F^{e_j} ’s.

We now conclude the proof. Say $\mathbf{x}' \in S^3(\mathbf{x})$ satisfies $F^{e_i}(\mathbf{x}') \in S^3(F^{e_i}(\mathbf{x}))$, and $[Z(\mathbf{x}') : V(\mathbf{x}'')] > 0$ (the case when $[Z(\mathbf{x}'') : V(\mathbf{x}')] > 0$ is similar). By the proof of the previous theorem, $[Z_j(x'_j) : V_j(x''_j)] > 0 \ \forall j$. Hence by the given assumption,

$$F^{e_i}(\mathbf{x}'') \in F^{e_i}(\times_j S'_j(x'_j)) = S'_i(F(x'_i)) \times (\times_{j \neq i} S'_j(x'_j)) \subset S^3(F^{e_i}(\mathbf{x}')),$$

where the last inclusion follows from Theorem 16.2. But then by choice of \mathbf{x}' , we have $F^{e_i}(\mathbf{x}') \in S^3(F^{e_i}(\mathbf{x}))$, so $F^{e_i}(\mathbf{x}'') \in S^3(F^{e_i}(\mathbf{x}))$ too.

The other relation is that of duality. But if $\mathbf{x}' \in S^3(\mathbf{x})$ satisfies $F^{e_i}(\mathbf{x}') \in S^3(F^{e_i}(\mathbf{x}))$, then $F^{e_i}(F(\mathbf{x}')) = F(F^{e_i}(\mathbf{x}')) \in S^3(F^{e_i}(\mathbf{x}))$, since this latter set is also closed under duality ($F = F^{(1, \dots, 1)}$).

Hence the closure under both these relations (of \mathbf{x} , which does satisfy: $F^{e_i}(\mathbf{x}) \in S^3(F^{e_i}(\mathbf{x}))$) also satisfies the same property. Thus, $F^{e_i}(S^3(\mathbf{x})) \subset S^3(F^{e_i}(\mathbf{x}))$ for all i , which finishes the proof of the claim. \square

17. FUNCTORIALITY OF THE BGG CATEGORY: COMBINING THE SETUPS

Combining all of the above analysis, we see that we have four setups in which it makes sense to consider the BGG Category \mathcal{O} , as given in (1.1). We now sketch a proof of Proposition 4.1 above:

That the first diagram commutes, is proved in Theorem 16.1; for the second diagram, use Proposition 16.2 and Corollary 16.1. The commuting cube in the \mathcal{C} 's uses the first diagram in Proposition 4.1, Proposition 8.2, Theorem 16.1, and Corollary 16.1. Finally, the result on Verma modules uses Remark 7.1 and Corollary 14.1. (The cube mentioned immediately after Proposition 4.1, also uses these results.)

We now show the main theorems, after stating a part of Theorem 4.2 in full detail:

Proposition 17.1. *Given $\lambda_i \in G_i$ for all i , the following are equivalent:*

- (1) $V_{A_i}(\lambda_i)$ is finite-dimensional for all i .
- (2) $V_A(\lambda)$ is finite-dimensional, where $\lambda = (\lambda_1, \dots, \lambda_n)$.
- (3) $V_{A_i \rtimes \Gamma_i}(x_i)$ is finite-dimensional for all i and any x_i with $\lambda_i = \lambda_{x_i}$.
- (4) $V_{A \rtimes \Gamma}(\mathbf{x})$ is finite-dimensional for any \mathbf{x} with $\lambda = \lambda_{\mathbf{x}}$.

and the following are equivalent:

- (1) $Z_{A_i}(\lambda_i)$ has finite length for all i .
- (2) $Z_A(\lambda)$ has finite length, where $\lambda = (\lambda_1, \dots, \lambda_n)$.
- (3) $Z_{A_i \rtimes \Gamma_i}(x_i)$ has finite length for any i and x_i with $\lambda_i = \lambda_{x_i}$.
- (4) $Z_{A \rtimes \Gamma}(\mathbf{x})$ has finite length for any \mathbf{x} with $\lambda = \lambda_{\mathbf{x}}$.

In all cases, V and Z stand for simple and Verma objects in the appropriate Category \mathcal{O} , respectively.

Proof. The first set of equivalences follows from Proposition 16.2 (for the vertical arrows in diagram (1.2)) and Theorem 7.1 (for the horizontal arrows). The second set of equivalences follows from Theorems 7.1 and 16.2. \square

Finally, we “collect together” a proof of the main results of this article.

Proof of Theorem 4.1. The first part is Theorem 16.1; the second comes from Propositions 10.2 and 11.3. For the third part, for $m = 3$ and 4, use Corollary 16.2 and Lemma 11.4 respectively. Given the result for $m = 3$, apply $\pi = \times_i \pi_i$ to get it for $m = 2$; moreover, this implies the $m = 1$ statement because of the partial order on G . \square

Proof of Theorem 4.2.

- (1) The equivalence of complete reducibility holding in the four setups, follows from Theorems 15.1 and 15.2. The second and third parts are shown above. The fourth part (about \mathcal{O} and Verma modules being of

finite length) follows from Proposition 9.2. Finally, the equivalences of the Conditions (S) across the four setups follow from Theorems 10.1 and 4.1 and Proposition 11.3.

(2) This follows from Theorem 10.1 and Proposition 11.3. \square

Proof of Theorem 4.3. The first part follows from Theorem 10.1, and the final two parts from Theorem 12.1, Corollary 12.1, and Proposition 12.2. \square

APPENDIX A. APPLICATION: SYMPLECTIC OSCILLATOR ALGEBRAS

A.1. Definitions. Now fix $A = H_f$ for a polynomial f (see [Kh1]) and $n \in \mathbb{N}$. (Note that H_f is an *infinitesimal Hecke algebra over \mathfrak{sl}_2* , as defined by Etingof, Gan, and Ginzburg in [EGG].) Define $H_{f,n} := (H_f)^{\otimes n} \rtimes S_n = S_n \wr H_f$. Then (we work here over $k = \mathbb{C}$, and) the analysis above yields:

Theorem A.1. *$H_{f,n}$ has a triangular decomposition. Moreover, if $1 + f \neq 0$, then \mathcal{O} is an abelian, finite length, self-dual category with block decomposition, as above. Each block is a highest weight category.*

This is because $A = H_f$ is then a strict Hopf RTA that satisfies Condition (S3), from [Kh1]. Moreover, from [KT], $CC_A(\lambda)$ is finite for all $\lambda \in G$, since the center is “large enough”. We can also characterize all finite-dimensional simple modules in \mathcal{O} , by [Kh1, Theorem 11], and the values of f for which complete reducibility holds in \mathcal{O} (see [GK, Theorem 10.1]).

A.2. Digression: Deforming a smash product. Suppose a Lie algebra \mathfrak{g} acts on a vector space V_0 . One defines a Lie algebra extension $V_0 \rtimes \mathfrak{g}$, via:

$$[v, v'] := 0 \quad [X, v] := X(v) \quad \forall v, v' \in V_0, X, X' \in \mathfrak{g}$$

Now suppose that $V, V' \subset V_0$ are finite-dimensional \mathfrak{g} -submodules, and we want to deform the relations $[v, v']$ (inside the algebra $\mathfrak{U}(V_0 \rtimes \mathfrak{g})$), with desired images in $\mathfrak{U}\mathfrak{g}$. In particular, $[X, [v, v']]$ should equal $\text{ad}(X)([v, v']) \in \mathfrak{U}\mathfrak{g}$. Thus we need a \mathfrak{g} -invariant element ω of $\text{Hom}_k(V \wedge V', \mathfrak{U}\mathfrak{g})$, i.e., $\omega \in \text{Hom}_{\mathfrak{g}}(V \wedge V', \mathfrak{U}\mathfrak{g}) = ((V \wedge V')^* \otimes_k \mathfrak{U}\mathfrak{g})^{\mathfrak{g}}$. In particular, if $V = V'$ has dimension 2, then $\omega \in (k \otimes_k \mathfrak{U}\mathfrak{g})^{\mathfrak{g}} = \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$, so $[v, v']$ must be central.

Now apply this to $H_{f,n}$, where \mathfrak{sl}_2 acts on $V = kX \oplus kY$ inside each copy $(H_f)_i$. Hence any deformed relations $[Y_i, X_i]$ must be of the form $f(\Delta_i)$. Moreover, the element $\sum_i [Y_i, X_i]$ must be central in the skew group ring $\mathfrak{U}\mathfrak{g} \rtimes S_n$ (for $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$), for it is central in $\mathfrak{U}\mathfrak{g}$ as above, and it is invariant under any transposition, hence under S_n .

A.3. Certain other deformations do not preserve the triangular decomposition. We consider certain deformations of $H_{f,n}$ and show that they do not preserve the triangular decomposition. More precisely, consider relations of the form

$$[Y_i, X_j] = \begin{cases} f(\Delta_i) + \sum_{l \neq i} (cs_{il} + d)(m(1 \otimes S)\Delta_{il})(\Delta) & \text{if } i = j \\ us_{ij} + vs_{ij}(m(1 \otimes S)\Delta_{ij})(\Delta) + w_{ij} & \text{if } i \neq j \end{cases} \quad (\text{A.1})$$

where

$c, d, u, v \in k$, and $w_{ij} \in \mathfrak{U}\mathfrak{g}$ for all $i \neq j$ (where $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$),

m is the multiplication map : $H_{f,n} \otimes H_{f,n} \rightarrow H_{f,n}$,

Δ_{ij} is the comultiplication in $\mathfrak{U}(\mathfrak{sl}_2)$, with image in $\mathfrak{U}(\mathfrak{sl}_2)_i \otimes_k \mathfrak{U}(\mathfrak{sl}_2)_j \subset H_{f,n}$,

S is the Hopf algebra antipode map on $\mathfrak{U}(\mathfrak{sl}_2^{\oplus n})$, taking $X \in \mathfrak{sl}_2^{\oplus n}$ to $-X$,

and Δ is the Casimir element in $\mathfrak{U}(\mathfrak{sl}_2)$.

(Note that together with this subalgebra $\mathfrak{U}\mathfrak{g}$ comes its *Cartan subalgebra* $\mathfrak{h} = \oplus \mathfrak{h}_i$, where $\mathfrak{h}_i = k \cdot H_i \subset (H_f)_i$. Moreover, $H_f^{\otimes n}$ is \mathfrak{h} -semisimple.)

We consider these relations because they are similar to those found in certain wreath-product-type deformations (see [EM]). Here, we find symplectic reflections of the form $s_{ij}\gamma_i\gamma_j^{-1}$, which equals $s_{ij}(m(1 \otimes S)\Delta_{ij})(\gamma)$ under the Hopf algebra structure (note that $\gamma_i = f_i(\gamma)$ here). However, we would still like our deformations to have the triangular decomposition.

Theorem A.2. *The only deformations of the type in equation (A.1), that also have the triangular decomposition, occur when c, d, u, v, w_{ij} are all zero.*

Proof. Check that $(m(1 \otimes S)\Delta_{ij})(\Delta) = \Delta_i + \Delta_j + (e_i f_j + f_i e_j + (h_i h_j)/2)$. Now note that X_j and Y_i are all weight vectors for $\text{ad } \mathfrak{h}$; hence so also must be their commutator. Suppose $X_j \in (H_{f,n})_{\eta_j}$ and $Y_i \in (H_{f,n})_{-\eta_i}$ for all i, j . Then we should get $[Y_i, X_j] \in (H_{f,n})_{\eta_j - \eta_i}$ for all i, j .

Let us denote $e_i f_j + f_i e_j + (h_i h_j)/2$, by $m_{ij} = m_{ji}$. We now check the relations for $i = j$, for we know that $\sum_i [Y_i, X_i]$ must be central in $\mathfrak{U}\mathfrak{g} \rtimes S_n$:

$$\sum_i [Y_i, X_i] = \sum_i f(\Delta_i) + \sum_{i \neq j} (cs_{ij} + d)(\Delta_i + \Delta_j + m_{ij})$$

The first term is obviously central. We now use that the commutator of e_k with this sum is zero (for fixed k), to show $c = d = 0$. This commutator equals (up to scalar multiples)

$$\sum_{i \neq k} \left([e_k, cs_{ik}](\Delta_i + \Delta_k + m_{ik}) + (cs_{ik} + d)[e_k, \Delta_i + \Delta_k + m_{ik}] \right)$$

The first term equals $cs_{ik}(e_i - e_k)(\Delta_i + \Delta_k + m_{ik})$ and the second term is $(cs_{ik} + d)[e_k, m_{ik}]$, so we get that $[e_k, \sum_i [Y_i, X_i]]$ equals

$$= \sum_{i \neq k} cs_{ik}(e_i - e_k)(\Delta_i + \Delta_k + m_{ik}) + (cs_{ik} + d)(e_i h_k - h_i e_k)$$

To satisfy the triangular decomposition, we must have $c = 0$ (consider the coefficient of $s_{ik}f_i e_i^2$, in $cs_{ik}e_i \Delta_i = cs_{ik} \Delta_i e_i$), and hence $d = 0$.

Next, when $i \neq j$, we need $[Y_i, X_j]$ to be an $\text{ad } \mathfrak{h}$ -weight vector of weight $\eta_j - \eta_i$. But $w_{ij} \in \mathfrak{U}(\mathfrak{sl}_2^{\oplus n})$, and $\mathfrak{U}(\mathfrak{sl}_2^{\oplus n})_{\eta_j - \eta_i} = 0$ if $i \neq j$, since each weight space has weight in $\oplus_i \mathbb{Z}(2\eta_i)$. Similarly, the $\text{ad } h_i$ -actions on $[Y_i, X_j]$ and the other terms do not agree:

$$[h_i, s_{ij}(u + vm_{ij})] = s_{ij}(h_j - h_i)(u + vm_{ij}) + vs_{ij}[h_i, m_{ij}]$$

and thus the LHS is not a weight vector, unless $u = v = 0$ as well. \square

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